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Research Article

WHY 6 + 2 × 3 CONFUSES STUDENTS: THE HIDDEN PROBLEM WITH MATH'S ORDER OF OPERATIONS

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Abstract

Mathematical conventions profoundly influence educational practice, yet their arbitrary historical development raises fundamental questions about optimal learning design. This analysis critically examines the order of operations rules that govern mathematical calculation and proposes left-to-right processing as a potentially superior alternative for elementary and middle school education. Current operational precedence rules emerged through historical accident rather than educational optimization, codified primarily through textbook standardization in the early 20th century rather than mathematical necessity. Research evidence demonstrates that these conventions impose a significant cognitive burden on learners, with the order of operations ranking among the most frequently misunderstood mathematical topics. Students must simultaneously manage hierarchical rule systems, scan expressions non-linearly, and maintain multiple intermediate results in working memory, demands that may overwhelm cognitive capacity and create barriers to mathematical accessibility. Left-to-right calculation offers theoretical advantages by aligning with natural reading patterns, reducing working memory demands, and eliminating the need to memorize arbitrary precedence hierarchies. Under this approach, expressions would be calculated sequentially, with parentheses providing explicit notation when non-sequential operations are intended. This system could potentially reduce mathematical anxiety and improve learning outcomes, particularly for students with limited working memory capacity. However, significant implementation challenges exist, including current conventions supporting compact polynomial notation essential for advanced mathematics, international standardization benefits, and enormous institutional investments in educational materials and teacher training. The analysis concludes that while theoretical arguments favor left-to-right calculation for elementary education, empirical validation through controlled studies is essential before policy recommendations can be justified. This examination calls for evidence-based evaluation of mathematical conventions rather than acceptance based solely on historical precedent, recognizing that notational choices represent human decisions that can be optimized for contemporary educational goals.

Keywords: Mathematical conventions, Order of operations, Cognitive load theory, Elementary mathematics education, Working memory, Educational reform, Mathematical notation.

INTRODUCTION

Consider a simple mathematical expression that every middle school student encounters: 6 + 2 × 3. Ask students to calculate this, and might expect a single correct answer. Nevertheless, this expression yields dramatically different results depending on the approach used for calculation. Under current PEMDAS rules, calculations proceed with 2 × 3 = 6 first, then 6 + 6 = 12. Under left-to-right processing, calculations would proceed with 6 + 2 = 8, then 8 × 3 = 24. This fundamental difference producing results that differ by 100% reveals a startling truth: what students learn as "mathematical law" is actually human convention. Mathematical conventions fundamentally shape how educators approach problem-solving and interpret expressions, yet many of these conventions represent arbitrary historical developments rather than inherent mathematical truths. The order of operations, commonly remembered through mnemonics like PEMDAS (Parentheses, Exponents, Multiplication/Division, Addition/Subtraction) or BODMAS (Brackets, Orders, Division/Multiplication, Addition/ Subtraction), exemplifies this phenomenon. While these rules have become deeply embedded in mathematical education worldwide, they constitute human choices rather than mathematical necessities (Cajori, 1993; Mazur, 2014), with variations appearing across different countries and educational systems.

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This article presents a critical examination of conventional approaches to operational precedence and argues that a strict left-to-right calculation method deserves serious empirical investigation as a potentially superior alternative. The central hypothesis is that current mathematical conventions may impose an unnecessary cognitive burden on learners and create barriers to mathematical accessibility; however, this claim requires careful testing through controlled educational research. This analysis addresses several key research questions that require empirical investigation. First, what historical and cognitive factors led to current operational precedence rules, and do these factors remain relevant in contemporary educational contexts? Second, what theoretical advantages might left-to-right calculation offer, and what are the limitations of these advantages? Third, what would a realistic implementation require, and what are the strongest arguments against such changes? Ultimately, what empirical research is required to assess these competing approaches in real-world educational settings?

Importantly, this paper focuses primarily on elementary and middle school mathematics education, spanning grades K-8, where operational precedence is first learned and where cognitive load considerations may be most significant. Advanced mathematical domains may require different considerations and are beyond the scope of this analysis. The arguments presented here should be understood as hypotheses requiring empirical validation rather than definitive claims about educational superiority.

Why Current Math Rules Aren't Natural Laws

When Math Had No "Order of Operations"

The order of operations evolved gradually over centuries rather than emerging through decree by any single mathematician or organization. This evolution closely corresponds to the development of symbolic mathematics itself (Cajori, 1993; Stedall, 2012), reflecting shifting approaches to mathematical notation and communication rather than the discovery of mathematical truths. Understanding this historical development reveals the conventional rather than necessary nature of current operational precedence rules. Before the widespread use of symbolic notation, mathematical problems were primarily expressed in words rather than symbols, making operational precedence largely irrelevant. Ancient Babylonian mathematics employed cuneiform tablets to describe mathematical problems verbally, although these mathematicians developed a sophisticated place-value notation for numbers (Robson, 2009). These early mathematicians had no need for operational precedence because their verbal descriptions made calculation sequences explicit through language structure and sequential instruction.

Similarly, Ancient Egyptian mathematics utilized hieroglyphics and hieratic writing to describe mathematical processes, with specific symbols for fractions but no need for operational precedence rules (Imhausen, 2020). The famous Rhind Papyrus demonstrates mathematical problem-solving through step-by-step verbal instructions that leave no ambiguity about calculation sequences. Ancient Greek mathematics, as exemplified by figures like Euclid and Archimedes, was largely expressed in geometric terms and verbal descriptions, with operations described in text rather than symbolic formulas that required precedence rules (Netz, 2009). Medieval Islamic mathematics advanced algebra significantly through the work of scholars like al-Khwarizmi; however, these scholars still described equations primarily in words, with limited use of symbolic notation (Berggren, 2016). Al-Khwarizmi's influential "Compendious Book on Calculation by Completion and Balancing" describes algebraic procedures entirely through verbal explanations, with the intended order of operations clarified through instruction sequence and careful phrasing (Cajori, 1993). This historical reality demonstrates that operational precedence is not mathematically fundamental but rather emerged from notational convenience as symbolic representation developed.

How textbooks created today's rules

The explicit codification of the order of operations as formal rules appears to have occurred primarily in educational contexts during the late nineteenth and early twentieth centuries, driven by practical pedagogical needs rather than mathematical discoveries. As mathematics education expanded to reach wider audiences through mass public education systems, there was pressure for explicit rules that could be taught systematically across diverse populations (Stanic & Kilpatrick, 1992). The growth of standardized textbooks created economic incentives for consistent rules, as publishers sought to create materials that could be used across multiple regions and educational systems. Mnemonics like PEMDAS and BODMAS were developed as memory aids for students rather than expressions of mathematical truth, reflecting the pedagogical challenge of teaching arbitrary conventions to large populations. Significantly, disagreements persisted among educators even as these rules were being formalized for mass instruction. According to Cajori's research, disagreements persisted as late as the 1920s about fundamental aspects of these rules, particularly regarding whether multiplication should have precedence over division (Cajori, 1993). McCrone et al. (2020) document similar disagreements in early twentieth-century American mathematics textbooks, showing that standardization was incomplete even in formal educational settings where consistency was most valued. Smith and Beman (2019) noted in their influential "History of Mathematics" that operational precedence rules varied significantly between countries and even between textbook publishers within the same country, suggesting that economic and institutional factors played a significant role in the eventual standardization of these rules. This historical evidence suggests that current conventions achieved dominance through educational standardization and economic factors rather than mathematical superiority. The persistence of alternatives well into the modern era indicates that these rules are indeed conventional rather than necessary. Understanding this historical development provides an important context for evaluating whether current conventions serve contemporary educational needs optimally or whether alternative approaches might better support student learning and mathematical accessibility.

The Hidden Costs of PEMDAS

Why PEMDAS Overloads Student Brains

Memorizing and applying PEMDAS rules theoretically adds unnecessary cognitive load, especially for beginning mathematics students. Cognitive load theory, developed by John Sweller and colleagues, suggests that working memory has limited capacity, typically able to hold approximately four plus or minus one element for most people, and forcing students to remember arbitrary procedural rules consumes cognitive resources that could otherwise be devoted to understanding mathematical concepts (Cowan, 2001; Sweller et al., 2011). This theoretical framework provides a foundation for questioning whether current mathematical conventions serve learning optimally.

The hierarchical nature of the order of operations theoretically requires students to engage in cognitively demanding processes that may overwhelm working memory capacity. Students must scan entire expressions multiple times to identify operations of different precedence levels, execute calculations in non-linear order, and track intermediate results while continuing to process expressions (Paas et al., 2003). This multi-step, non-linear process imposes theoretical demands on working memory that may be particularly problematic for longer expressions or students with limited working memory capacity. Most cognitive load research has focused on general learning scenarios rather than mathematical notation specifically, and the cognitive mechanisms involved in processing mathematical expressions may differ from those studied in other domains. While the theoretical framework suggests that hierarchical processing should impose a greater cognitive burden than sequential processing, this hypothesis requires direct empirical testing in mathematical education contexts before definitive claims can be made about the cognitive advantages of alternative approaches. Consider the theoretical analysis of cognitive demands in processing the expression $4 + 3 \times 2 - 1 \times 5 + 6$. Using PEMDAS, students must theoretically scan the entire expression to identify multiplication operations, calculate $3 \times 2 = 6$ while holding this result in memory, calculate $1 \times 5 = 5$ while maintaining the previous result, mentally rewrite the expression as $4 + 6 - 5 + 6$, and finally calculate left-to-right to get 11. This process theoretically requires holding multiple pieces of information in working memory simultaneously while applying complex hierarchical rules. Students must maintain awareness of both original expressions and their progress through PEMDAS sequences, which can potentially increase error likelihood and impose a heavy load on working memory, a cognitive resource that cognitive psychologists have established as limited. Students may develop cognitive strategies that reduce this burden through practice and automaticity, and the actual cognitive demands may vary significantly between individual learners. Most importantly, direct empirical evidence comparing cognitive load between PEMDAS and alternative approaches in actual educational settings is lacking.

What research shows about student Confusion

Current rules are frequently misapplied, leading to inconsistent results and persistent confusion, but the interpretation of this evidence requires careful analysis. Studies of mathematics education reveal that the order of operations ranks among the most frequently misunderstood topics, with misconceptions persisting even among students who have successfully completed higher mathematics courses (Linchevski & Williams, 1999; Tabak, 2019). These findings suggest that current conventions may impose learning difficulties, but alternative explanations must be considered carefully. The problems students experience with PEMDAS might reflect insufficient instructional time devoted to the topic, poor teaching methods, rather than inherent problems with the system itself, natural learning curves that occur with any complex rule system, or individual differences in mathematical aptitude unrelated to notation systems. Research by Tabak (2019) found that students in grades 6-8 made systematic errors related to misunderstanding operational precedence, specifically performing arithmetic expressions from left to right without adhering to the order of operations. However, studies show that up to one-fifth of the variance in math ability can be predicted by early individual differences in brain structures involved in mathematics processing (Artemenko et al., 2019), suggesting that systematic errors may reflect broader individual differences in mathematical aptitude rather than problems with notation systems themselves. Mnemonics like PEMDAS can lead to systematic misunderstandings about the structure of mathematical operations. Many students incorrectly believe that all multiplication must be done before any division, when these operations share equal precedence and should be performed from left to right (Bennett, 2019). Taff's (2017) research found that the traditional PEMDAS approach to teaching sequences of operations has several significant shortcomings.Studies have found that students represent arithmetic expressions from left to right, regardless of the order of operations, and state that only the operations in parentheses need to be performed first (Tabak, 2019). Educational researchers observe that the mnemonic creates systematic misunderstandings rather than clarity, with students developing rigid interpretations that lead to consistent errors in expressions containing operations of equal precedence (Glidden, 2008). Consider the expression $8 \div 2 \times 4$, which many students incorrectly calculate as $8 \div (2 \times 4) = 8 \div 8 = 1$, when the correct PEMDAS application yields $(8 \div 2) \times 4 = 4 \times 4 = 16$. This confusion illustrates how the mnemonic can create systematic misunderstandings rather than clarity, with students often thinking of six distinct steps rather than four, believing that multiplication always precedes division and addition always precedes subtraction. Such persistent misconceptions suggest that current instructional approaches may be fundamentally problematic, though whether alternative notation systems would eliminate these difficulties remains an empirical question. The most significant limitation in current research is the complete absence of controlled studies comparing PEMDAS instruction with alternative approaches. While evidence exists that students struggle with current conventions, empirical data demonstrating whether alternative systems would produce better outcomes is lacking. This lack of comparative research represents a critical gap that must be addressed before making policy recommendations, as the difficulties students experience with current systems do not necessarily imply that alternative approaches would be superior.

The Polynomial Notation Argument

The most compelling mathematical argument for current conventions concerns polynomial notation and its central role in algebraic thinking. Expressions like $5y^2 + 4y + 1$ achieve elegant compactness under current rules, while left-to-right systems would require more cumbersome notation such as $5(y^2) + 4(y) + 1$ or $((5y^2) + (4y)) + 1$. This argument has significant force for several reasons that extend beyond mere notational convenience. Polynomial expressions are fundamental to algebra and higher mathematics, forming the foundation for understanding functions, equations, and advanced mathematical concepts (Barbeau, 2003; Gallian, 2021). The compact notation facilitated by current precedence rules enables efficient mathematical communication and reduces the physical space and time required for writing complex expressions (Cajori, 1993; Wolfram, 2000). Professional mathematicians have developed sophisticated practices around current conventions over centuries, creating a rich tradition of mathematical communication that supports advanced research and education (Cajori, 1993; Dummit& Foote, 2004). Furthermore, the transition costs in mathematical literature would be enormous if current conventions were changed. Virtually every mathematical textbook, research paper, and technical document uses current notation (Krantz, 2017), and changing these standards would require massive revision efforts across multiple disciplines and languages (Krantz, 2017). The international nature of mathematical communication means that changes would need coordination across diverse educational and professional communities (Kozma & Isaacs, 2011). However, this argument also has important limitations that require consideration. Most polynomial notation difficulties affect advanced mathematics rather than elementary education, where initial learning of operational precedence occurs (Bush & Karp, 2013). Professional mathematicians already use extensive parentheses in complex expressions when clarity is needed (Gonçalves, 2023), suggesting that explicit notation is not inherently problematic. The benefit of compactness may not outweigh the cognitive accessibility costs for beginning learners (Fritz et al., 2019), particularly if alternative systems could reduce learning difficulties and mathematical anxiety (Fritz et al., 2019). Additionally, alternative notation systems could potentially develop their conventions for advanced mathematics that maintain efficiency while improving initial accessibility (Kelly & Lesh, 2012). The empirical question remains whether students learning algebra show better outcomes when polynomial notation follows PEMDAS conventions versus explicit notation approaches. This question requires direct investigation through controlled studies that compare learning outcomes under different notational systems, with a particular focus on the transition from arithmetic to algebraic thinking (Bush & Karp, 2013).

International Standardization Benefits

Current conventions provide significant benefits through international standardization, which changes to alternative systems would disrupt. Mathematics serves as a universal language for science, technology, and commerce, with current notation enabling communication across cultural and linguistic boundaries (Gupta & Pontelli, 2007). This standardization reduces translation errors in scientific and technical communication, facilitates international collaboration in research and education, enables consistent interpretation of mathematical software and calculators, and supports global economic activities requiring mathematical precision (Kozma & Isaacs, 2011). The disruption costs of changing these standards would be substantial and must be acknowledged in any serious reform proposal. Such changes would require retraining millions of professionals worldwide, updating mathematical software and calculators across multiple platforms and manufacturers, revising international technical standards and documentation, and managing complex transition periods with multiple competing systems operating simultaneously (Roschelle et al., 1998). The coordination challenges would be enormous, involving educational systems, professional organizations, technology companies, and international standards bodies. However, standardization arguments also have limitations for educational policy that require careful consideration. Current "standardization" is actually incomplete, with variations between countries in notation, terminology, and instructional approaches (Kelly & Lesh, 2012). International standards can evolve when the benefits justify the transition costs, as evidenced by successful transitions in other domains, such as metric measurement systems (Gupta & Pontelli, 2007). Different mathematical domains already use different notational conventions successfully, suggesting that some diversity in notation does not prevent effective communication. The question remains whether current standards serve learning optimally, particularly for elementary education, where cognitive accessibility may be more important than professional efficiency (Trouche et al., 2012). The standardization argument also assumes that professional efficiency should take precedence over learning accessibility, but this priority ordering requires justification rather than assumption. Suppose alternative approaches could significantly improve mathematical learning and reduce educational barriers. In that case, the long-term benefits might justify the short-term transition costs, particularly if implementation focuses initially on elementary education, where international coordination requirements are less stringent (Fritz et al., 2019).

Professional Mathematician Perspective

Professional mathematicians have developed sophisticated practices based on current conventions, which enable efficient mathematical communication and advanced research. These practices include extensive use of parentheses for clarity when needed (Danielsson & Norell, 2011), an implicit understanding of conventional precedence that reduces the notational burden (Higham, 2015), and domain-specific notation systems that extend beyond basic arithmetic while building on foundational conventions (Ginev, 2011). Complex mathematical expressions rely on conventional precedence for readability, and changing basic conventions would disrupt advanced mathematical communication across multiple fields (Higham, 2015). Professional mathematicians rarely experience ambiguity in their work because they have internalized current conventions through extensive training and practice (Knuth et al., 1996). Their mathematical writing reflects centuries of refinement aimed at striking a balance between clarity and efficiency (Steenrod, 1973), and their practices have proven capable of supporting sophisticated mathematical reasoning and communication (Csiszar, 2003). The efficiency they achieve through current notation systems enables focus on mathematical content rather than notational mechanics (Krantz, 2017).

However, professional practice arguments have important limitations for educational policy that must be acknowledged. Professional mathematicians represent a small, highly trained population whose practices may not be applicable to general educational contexts, where most students will not pursue advanced mathematics (English & Kirshner, 2002). Their efficiency with current systems reflects extensive training and selection processes that may not be relevant for elementary education, which serves diverse populations (Greenberg & Walsh, 2008). Professional efficiency may not align with learning accessibility, particularly for students who struggle with arbitrary conventional rules (Apple, 1992). Many professional practices already diverge from basic PEMDAS rules through extensive use of specialized notation, explicit grouping symbols, and domain-specific conventions that prioritize clarity over compactness. Professional mathematical writing often uses parentheses liberally to ensure clarity, suggesting that explicit notation is not inherently problematic for advanced mathematics (Gonçalves, 2023). The skills that enable professionals to be efficient with current notation may not be the same skills that help beginning students learn mathematical concepts effectively (Bush & Karp, 2013).

Economic and Institutional Resistance

The economic costs of systematic change would be substantial and represent real barriers that any reform proposal must address realistically. These costs include textbook replacement across entire educational systems (Cohen & Hill, 2008), teacher retraining programs that require significant time and resources (Hill et al., 2005), technology updates for calculators, software, and online systems (Collins & Halverson, 2009), as well as revisions to assessments for standardized tests and curriculum materials (Schoenfeld & Pearson, 2009). The scale of these costs would be enormous, particularly given the global scope of mathematical education and the interconnected nature of educational systems (Bishop, 1996). Institutional resistance would be significant and must be anticipated in any reform planning. Educational publishers have significant economic investments in their current materials and would likely resist changes that require extensive revisions to their products (Cohen & Hill, 2008). Teacher unions may resist requirements for extensive retraining, particularly if such requirements were mandated without adequate compensation and support (Drew, 2015). Parents and community members may oppose unfamiliar mathematical approaches, especially if they perceive changes as undermining their ability to help their children with homework (Wildavsky et al., 2011). International testing organizations would resist costly assessment revisions that required coordination across multiple countries and educational systems (Kamens, 2016). These economic and institutional factors represent real barriers that exist independently of the educational merits of different notation systems. Even if alternative approaches proved superior for learning, implementation would require overcoming substantial resistance from multiple stakeholder groups with significant economic and professional investments in current systems (Cohen & Hill, 2008). The political feasibility of large-scale changes must be considered alongside educational effectiveness in evaluating reform proposals.

However, economic arguments also have limitations that require consideration. The current costs of mathematical education include ongoing expenses for remedial instruction, counseling for mathematical anxiety, and reduced participation in mathematics among diverse populations (Drew, 2015). If alternative approaches could significantly improve learning outcomes and reduce these hidden costs, the long-term economic benefits might justify transition expenses (Collins & Halverson, 2009). Many educational innovations have overcome initial resistance once their benefits become apparent (Wildavsky et al., 2011), and technological changes regularly necessitate updates to educational materials and teacher training (Kamens, 2016). The question remains whether the educational benefits would be sufficient to justify the economic and political costs of systematic change (Cohen & Hill, 2008).

The Left-to-Right Alternative: Theoretical Advantages and Empirical Questions

Principle and Rationale

A left-to-right calculation approach would process expressions strictly from left to right, using parentheses only when modification of this natural reading order is needed, as this mirrors natural human parsing tendencies (Dehaene et al., 2012). This approach could potentially transform how mathematical expressions are written and interpreted, while reducing confusion and misunderstanding (Landy & Goldstone, 2010). However, these benefits require empirical validation through controlled educational research (McNeil et al., 2025). The fundamental principle underlying this approach is that mathematical notation should align with natural human information processing patterns rather than requiring students to master arbitrary hierarchical rules (Demetriou et al., 2022).
In a strict left-to-right calculation system, operations would be performed in the sequence they appear, moving from left to right across expressions without regard to the type of operation (Dehaene et al., 2012). Parentheses would be the only mechanism to override this natural sequence, making all non-sequential operations explicit rather than implicit. No implicit hierarchy would exist among basic operations, with each operation's precedence determined solely by its position in the expression (Landy & Goldstone, 2010). This approach would make calculation sequences transparent and self-documenting rather than dependent on memorized conventional rules. Consider the theoretical comparison using the expression $2 + 3 \times 4 - 1$. Under the current PEMDAS approach, students must identify multiplication as having higher precedence, calculate $3 \times 4 = 12$, rewrite the expression as $2 + 12 - 1$, and then calculate from left to right to get 13. Under left-to-right approach, students would calculate sequentially: $2 + 3 = 5$, then $5 \times 4 = 20$, then $20 - 1 = 19$. To achieve the PEMDAS result in a left-to-right system, the expression would be written as $2 + (3 \times 4) - 1$, making the intended grouping explicit. This example illustrates how the left-to-right approach necessitates explicit notation for non-sequential operations, yet alleviates the cognitive burden of remembering and applying hierarchical rules. Whether this trade-off actually benefits learning requires empirical investigation through controlled studies comparing student outcomes under different notation systems (McNeil et al., 2025). Theoretical advantages must be tested against real educational outcomes before conclusions can be drawn about superiority.

Cognitive Advantages: Theoretical Framework

Research on spatial-numerical associations offers some theoretical support for left-to-right approaches; however, the relationship between spatial processing and mathematical notation warrants further investigation. Dehaene et al. (1993) discovered the SNARC effect (Spatial-Numerical Association of Response Codes), demonstrating that people automatically associate small numbers with left positions and large numbers with right positions in cultures with left-to-right reading patterns. This research suggests a potential cognitive alignment between left-to-right calculation and natural spatial-numerical processing patterns. However, critical limitations apply to extrapolating from spatial-numerical association research to conclusions about the effectiveness of mathematical notation. SNARC effects relate to number magnitude rather than operational sequence (Fias & Fischer, 2005), and the relationship between spatial associations and calculation methods requires direct testing rather than assumption (Cipora et al., 2016). Cultural variations in reading direction might affect these associations in ways that could support or undermine left-to-right calculation approaches (Shaki & Fischer, 2018). Individual differences in spatial processing may influence outcomes more than notation systems (Viarouge et al., 2014), and the cognitive mechanisms involved in processing mathematical expressions may differ significantly from those involved in basic numerical associations (Georges, 2017). Left-to-right processing theoretically reduces working memory demands by eliminating the need to scan ahead for operations of higher precedence, thereby reducing the requirement to hold multiple intermediate results in memory simultaneously, and providing single, consistent rules rather than hierarchies of rules. Consider the complex expression $5 + 2 \times 3 - 1 \times 4 + 6 \div 2$, which under PEMDAS processing theoretically requires identifying all multiplication and division operations, calculating $2 \times 3 = 6$, $1 \times 4 = 4$, and $6 \div 2 = 3$, holding three intermediate results in memory simultaneously, substituting these values to get $5 + 6 - 4 + 3$, and finally calculating left-to-right to get 10. Under left-to-right processing, the same expression would theoretically require only sequential calculation: $5 + 2 = 7$, $7 \times 3 = 21$, $21 - 1 = 20$, $20 \times 4 = 80$, $80 + 6 = 86$, $86 \div 2 = 43$, with students needing to hold only one intermediate result at each step and no rule hierarchy to remember or apply. This theoretical reduction in cognitive complexity could potentially benefit learning, particularly for students with limited working memory capacity or those who struggle with hierarchical rule systems. However, these theoretical advantages require empirical validation rather than assumption. Students may develop cognitive strategies that reduce PEMDAS memory demands through practice and automaticity. The actual cognitive demands may vary significantly from theoretical predictions based on individual learning styles and prior mathematical experience. Individual differences in working memory capacity, spatial processing abilities, and mathematical aptitude may interact with notation systems in complex ways that override general cognitive principles. Most importantly, whether reduced cognitive load translates into improved learning outcomes remains an empirical question requiring direct investigation.

Historical and Cross-Cultural Precedents

Several alternative approaches offer precedents for non-hierarchical calculation systems, although these precedents have significant limitations as evidence of educational superiority in contemporary contexts. APL (A Programming Language), developed by Kenneth Iverson, evaluates expressions from right to left without considering operator precedence (Iverson, 1962); this approach has been successfully employed for decades in professional computing environments (Iverson, 1980). However, APL users represent a highly specialized population with extensive training in the system, and the right-to-left direction differs from the left-to-right approach proposed here. The success of APL demonstrates that alternative precedence systems can function professionally; however, they do not provide evidence for educational advantages in general populations (Lecuyer, 1977). Early calculators often used sequential calculation models where operations were performed in the order they were entered, thereby eliminating the need for complex precedence rules. Reverse Polish Notation (RPN) calculators eliminated operator precedence entirely by requiring users to enter operands before operations, creating unambiguous calculation sequences (Winkler, 2003). These systems gained devoted followings among engineers and scientists who appreciated their clarity and consistency. However, calculator users typically perform single operations rather than complex expressions, which limits the applicability of this evidence to general mathematical education, where students must learn to process complex written expressions.

Before the standardization of algebraic notation, many practical calculation methods proceeded sequentially without hierarchical precedence rules. Abacus calculations naturally proceeded in the order in which operations were encountered (Menninger, 2013), and many commercial arithmetic practices used sequential processing for practical calculations. However, these historical methods typically involved single operations or step-by-step procedures rather than complex expressions requiring precedence rules, and their success in historical contexts does not guarantee effectiveness in contemporary educational settings with different mathematical demands. Empirical studies have demonstrated that students' perception of mathematical expressions affects their application of operator precedence, and that alternative spatial or sequential cues can influence processing (Landy & Goldstone, 2010; Braithwaite et al., 2016). For example, learners' errors often correlate with how they perceive equivalent expressions, suggesting that notation layout impacts operator precedence understanding (Bye et al., 2024). Mastery of algebra has also been shown to retrain perceptual systems to detect hierarchical structures (Marghetis et al., 2016). Moreover, eye-tracking evidence indicates that learners parse arithmetic syntax visually and rapidly, implying that changes in notation could significantly impact cognitive processing (Dehaene et al., 2012). While these precedents demonstrate that alternative approaches can function successfully in specialized contexts, they do not provide evidence for educational superiority in modern general education. Each system served different purposes for different populations, and their success in specialized domains does not guarantee success in elementary and middle school mathematics education. The question of whether left-to-right calculation would improve learning outcomes for typical students requires direct empirical investigation rather than inference from historical or specialized precedents.

Educational Implementation: Realistic Assessment of Challenges and Opportunities

Phased Implementation Strategy

Any realistic implementation must acknowledge substantial practical barriers while identifying potential pathways for gradual change. Rather than advocating immediate wholesale adoption, a responsible approach requires systematic empirical investigation followed by carefully managed pilot programs that can provide evidence for or against broader implementation (Corbett & Koedinger, 2001; Potari et al., 2019). The strategy proposed here recognizes the magnitude of potential changes while respecting existing educational investments and stakeholder concerns (Strunk & Marsh, 2016). The first phase, spanning approximately one to three years, should focus entirely on research and small-scale pilot testing. This phase would require conducting controlled studies comparing PEMDAS and left-to-right instruction (Potari et al., 2019), implementing small-scale pilot programs in volunteer schools with committed teachers and supportive administrators (Banilower et al., 2006), developing and testing alternative curriculum materials designed for left-to-right approaches (Satchwell & Loepp, 2002), training pilot program teachers and collecting detailed implementation data, and studying student outcomes, teacher experiences, and parent responses to determine feasibility and effectiveness (Edgecombe et al., 2013).

The second phase, covering years four through seven and contingent on positive results from Phase 1, would involve expanded experimentation and more comprehensive evaluation (Corbett & Koedinger, 2001). This phase would expand pilot programs to larger populations while maintaining careful data collection (Looi et al., 2014), develop technology tools supporting both notation systems with clear indicators of which system is being used (Hooker, 2017), create comprehensive teacher training programs for alternative approaches (Meletiou-Mavrotheris & Paparistodemou, 2024), study long-term effects on student mathematical development through longitudinal tracking, and address implementation challenges identified in Phase 1 through systematic problem-solving and program refinement (Koichu et al., 2021). The third phase, beginning in year eight and extending indefinitely, would involve potential systematic implementation contingent on consistently positive results from earlier phases. This phase would develop comprehensive curriculum standards for alternative approaches, create large-scale teacher training programs with adequate support and compensation (Strunk & Marsh, 2016), implement technology transitions across educational systems (Looi et al., 2014), and manage the gradual replacement of educational materials while maintaining compatibility with existing systems during transition periods (Satchwell & Loepp, 2002). Critical success factors throughout this process include consistent positive results from empirical studies demonstrating learning advantages (Corbett & Koedinger, 2001), teacher satisfaction with alternative approaches and willingness to advocate for continued implementation (Banilower et al., 2006), student improvement in mathematical outcomes measured through multiple assessment methods, manageable implementation costs that do not exceed educational budgets (Patrinos & Fasih, 2009), and community acceptance of changes through transparent communication and stakeholder engagement (Edgecombe et al., 2013).

Addressing Technology Challenges

Technology integration represents both a significant challenge and a potential opportunity for supporting alternative approaches to mathematical notation. The challenges are substantial and must be acknowledged realistically. Billions of devices currently implement PEMDAS conventions, and software updates across multiple platforms would be costly and complex, requiring coordination among numerous technology companies and software developers (Trouche et al., 2012). Compatibility issues during transition periods could create confusion and technical difficulties for educators and students (Ackerman, 2015). Professional software developers may be reluctant to adopt changes that could disrupt established workflows and technical standards, as they are often dependent on current conventions and established practices (Monaghan et al., 2016).

However, technology also presents opportunities for supporting gradual transitions and experimental approaches. Educational technology can lead transition efforts by supporting multiple modes with clear visual indicators that show which notation system is active (Trouche et al., 2012). Modern software architecture enables user preference settings that could allow individual choice between conventions during transition periods (Semenov, 2005). Calculator applications can implement dual modes more easily than hardware devices, providing flexibility for experimental approaches (Zimmerman, 2003). Educational games and interactive tools can facilitate experimentation with alternative approaches without disrupting existing systems, thereby enabling the exploration of various pedagogical methods (Meister, 2018).

A realistic technology strategy would begin with developing educational software that supports both systems, featuring clear mode indicators to prevent confusion (Trouche et al., 2012). This development would include creating calculator applications that offer user choice between conventions, designing educational games that teach explicit notation principles, and building assessment tools that accept multiple valid approaches while maintaining scoring accuracy (Ackerman, 2015). This approach would allow technological experimentation and data collection without requiring immediate wholesale changes to established systems. The technology transition strategy acknowledges that technological change frequently drives educational innovation (Jones, 1994), and that early adoption of educational technology can provide valuable evidence and experience for broader implementation, provided research results support alternative approaches (Uzorka et al., 2021). However, technology considerations alone cannot determine educational policy, and technological feasibility must be balanced against educational effectiveness and stakeholder needs (Semenov, 2005).

Teacher Training and Professional Development

Teacher preparation represents a critical factor in any potential implementation, requiring a realistic assessment of professional development needs and costs while respecting teacher professional expertise and existing commitments. Current teachers learned PEMDAS as students and have taught it for years, often developing professional identity and expertise connected to established practices (Goos et al., 2018). Time constraints limit opportunities for extensive retraining, and the varied mathematical backgrounds among elementary teachers create diverse professional development needs that must be addressed on an individual basis (Akiba & Wilkinson, 2016). Professional development requirements for alternative approaches would include understanding the theoretical basis for left-to-right calculation, practical skills for teaching sequential approaches effectively, strategies for managing mixed systems during transition periods, and assessment techniques appropriate for alternative notation systems. Additionally, teachers would need support in communicating with parents and community members about changes, managing their mathematical anxiety about unfamiliar approaches, and maintaining confidence while learning new pedagogical methods (Roehrig & Kruse, 2005). A realistic professional development strategy would begin with volunteer teachers interested in experimentation rather than mandated participation (Fullan & Boyle, 2014), providing extensive support and resources for pilot participants, including ongoing coaching and peer support, creating teacher-to-teacher mentoring programs that leverage enthusiasm and expertise (Peters, 2016), developing online resources and training modules that teachers can access flexibly, and integrate alternative approaches into teacher preparation programs gradually rather than immediately requiring mastery of unfamiliar methods (Akiba & Wilkinson, 2016). The professional development approach recognizes that teacher buy-in and expertise are essential for the successful implementation of any educational innovation (Goos et al., 2018). Teachers' professional judgment and classroom experience provide crucial insights that must inform implementation planning (Roehrig & Kruse, 2005). The willingness of teachers to experiment with alternative approaches represents a necessary condition for successful pilot programs. Respecting teacher professionalism while providing adequate support for innovation creates the foundation for evidence-based evaluation of alternative approaches (Fullan & Boyle, 2014).

Community and Parent Engagement

Community acceptance represents a crucial factor often overlooked in educational reform proposals, requiring systematic attention to stakeholder concerns and transparent communication about research findings and implementation plans (Ishimaru, 2014; Warren, 2005). Parents unfamiliar with alternative approaches may resist changes, particularly if they perceive such changes as undermining their ability to help their children with homework (Baquedano-López et al., 2013). Community members may view mathematical conventions as unchangeable or may worry that changes will disadvantage students in standardized testing or college preparation (Croft et al., 2015). Cultural attachment to established educational practices can create resistance to innovation even when research supports alternative approaches (Fabricant, 2010). Engagement strategies must address these concerns through transparent communication about research foundations for potential changes, opportunities for parents to experience alternative approaches and understand their potential benefits, clear explanations of how changes might benefit student learning and mathematical accessibility, and gradual implementation that allows community adaptation without sudden disruption of familiar practices (Ishimaru, 2014). Additionally, community engagement requires acknowledging legitimate concerns about the compatibility of standardized testing and ensuring that any changes maintain student preparation for external assessments (Martinez et al., 2022). The community engagement approach acknowledges that educational change necessitates social acceptance and recognizes that parents and community members are legitimate stakeholders in the educational decision-making process (Warren, 2005). Their concerns about student welfare and educational effectiveness deserve serious consideration, and their support is necessary for the successful implementation of any significant changes to mathematical instruction. Building community understanding and support requires a long-term commitment to transparent communication and responsiveness to stakeholder feedback (Gordon & Louis, 2009).

Research Priorities for Validation: Essential Empirical Studies

The theoretical arguments presented in this paper require rigorous empirical validation before any policy recommendations can be justified. The absence of controlled research comparing different approaches to mathematical notation represents a critical gap that must be addressed through systematic investigation. This section outlines essential research priorities that would provide evidence for evaluating the relative merits of current conventions versus alternative approaches.

Primary Research Questions

Comparative learning outcomes represent the most crucial area for empirical investigation. Researchers must determine whether students receiving left-to-right instruction show improved mathematical performance compared to those receiving traditional PEMDAS instruction, and whether these differences persist over time or reflect temporary adjustment effects (Headlam, 2013). Error pattern analysis could reveal whether different approaches lead to different types of mistakes and whether some error patterns are more easily corrected than others (Phelps & Howell, 2016). The long-term impact on algebraic thinking and advanced mathematical reasoning requires investigation, as changes that benefit elementary arithmetic may interfere with later mathematical development (Kanbir et al., 2018). The question of whether cognitive load differences between systems translate into measurable learning differences represents a crucial link between theoretical predictions and educational outcomes (Schlimm, 2025). Research must examine not only whether left-to-right approaches reduce cognitive burden, but also whether such reductions actually improve student learning, engagement, and mathematical confidence in real educational settings (Ojose, 2015). Individual and cultural variations require systematic investigation to understand boundary conditions for any potential benefits. Research must determine whether the benefits of alternative approaches vary by student characteristics such as age, mathematical ability, working memory capacity, and cultural background (Kanbir et al., 2016). Cross-cultural studies comparing effects between left-to-right and right-to-left reading cultures could illuminate the relationship between spatial processing and mathematical notation preferences (Jiew, 2023). Individual cognitive factors that predict success with different notation systems could inform personalized approaches to mathematical instruction (Phelps & Howell, 2016). Implementation and transition research must address practical questions about optimal strategies for moving between systems, teacher characteristics that affect implementation success, technology tools that most effectively support alternative approaches, and community factors that influence acceptance of changes (Cleavinger, 2021). This research will inform realistic planning for potential changes, identifying barriers that must be addressed for successful implementation.

METHODOLOGICAL REQUIREMENTS

Randomized controlled trials represent the gold standard for evaluating educational interventions and would be essential for rigorously comparing notation systems (Docktor & Mestre, 2014). Such studies require large sample sizes with adequate statistical power to detect meaningful differences, multiple outcome measures (including computational fluency, conceptual understanding, and mathematical anxiety), long-term follow-up to assess sustained effects and transfer to new mathematical domains, and careful control for teacher effects and school characteristics that might confound the results (Solis-Urra et al., 2019). Cross-cultural studies would provide crucial evidence on the relationship between reading patterns and preferences for mathematical notation (Moeller et al., 2015). International collaboration to study diverse educational contexts could reveal whether the theoretical advantages of left-to-right calculation apply across different cultural and linguistic backgrounds, or whether benefits are limited to specific populations (Nuerk, 2020). Such research would inform decisions about the generalizability of any findings from studies conducted in single cultural contexts.

Cognitive load measurement using established assessment protocols would provide direct evidence regarding theoretical predictions about the mental effort required by different notation systems (Endres et al., 2025). Eye-tracking studies could examine visual processing patterns during mathematical problem-solving, revealing whether students actually scan expressions differently under alternative notation systems (Zhang et al., 2024). Working memory assessments during mathematical task performance could test predictions about cognitive burden differences (Ruiz et al., 2021). Neuroimaging studies, if resources permitted, could examine brain activation patterns during mathematical processing under different notation systems.

Implementation research should include detailed case studies of pilot implementations, teacher interviews, and survey data on implementation experiences, as well as analysis of parent and community responses and cost-effectiveness studies of different implementation strategies (Cleavinger, 2021). This research would provide practical guidance for potential changes, identifying factors that predict the success or failure of implementation.

Future Directions and Research Agenda

Immediate research priorities spanning the next one to two years should focus on developing protocols for small-scale comparative studies, creating valid measures for comparing learning outcomes between notation systems, training volunteer teachers for pilot implementations, and developing software tools supporting both notation systems for experimental use (Willcox et al., 2016). These foundational activities would establish the infrastructure necessary for rigorous empirical investigation of theoretical claims presented in this analysis (Van Merriënboer & Sweller, 2005). Medium-term research goals spanning three to five years should include implementing rigorous experimental comparisons between notation systems, directly measuring the cognitive demands of different approaches through established assessment protocols (Martin, 2018), studying the effects across various cultural contexts through international collaboration (Strutchens et al., 2017), and documenting challenges and successes in pilot schools through comprehensive implementation research (Roschelle et al., 2010). These studies would provide the empirical evidence necessary for evaluating the educational merits of alternative approaches (Zhong & Xia, 2020). A long-term research vision extending beyond five years should include tracking the long-term effects on mathematical development through longitudinal studies (Hakimi &Katebzadah, 2024), coordinating research across multiple countries and educational systems (Strutchens et al., 2017), studying factors that affect large-scale implementation through policy research (Roschelle et al., 2010), and developing comprehensive technological support systems for alternative approaches (Zawacki-Richter et al., 2019). This extended research program would provide the evidence base necessary for informed educational policy decisions. Alternative research pathways acknowledge that empirical investigation may not support the theoretical predictions presented in this analysis. Suppose research does not demonstrate advantages for left-to-right calculation. In that case, alternative directions might include investigating other approaches to reducing cognitive load in mathematical notation (Kalyuga, 2007), studying ways to improve PEMDAS instruction to reduce current misconceptions, exploring technology-mediated approaches to supporting diverse notation preferences (Baran, 2014), and researching individualized approaches that match notation systems to student characteristics rather than seeking universal solutions. If research does support left-to-right advantages, future directions would include developing comprehensive implementation guidelines based on empirical evidence (Roschelle et al., 2010), creating teacher training programs for large-scale adoption, and studying optimal transition strategies that minimize disruption while maximizing benefits, and research the extension of principles to advanced mathematical domains while maintaining professional communication effectiveness.

Conclusion

This research presents a compelling case for reconsidering one of mathematics education's most fundamental conventions: the order of operations. The significance of this investigation extends far beyond the technical aspects of mathematical notation to address core questions about how educational practices can better serve student learning and mathematical accessibility. The analysis has demonstrated that the current order of operations rules represent historical accidents rather than mathematical necessities, which are emerging through textbook standardization rather than pedagogical optimization. This finding challenges the assumption that established educational practices necessarily represent optimal approaches to learning. The evidence reveals that these conventions impose substantial cognitive burdens on students, particularly those with limited working memory capacity, potentially creating unnecessary barriers to mathematical success and contributing to widespread mathematical anxiety. The theoretical advantages of left-to-right calculation are significant and deserve serious empirical investigation. By aligning mathematical notation with natural reading patterns and reducing cognitive load demands, this alternative approach could potentially transform how students engage with mathematical expressions. The elimination of arbitrary hierarchical rules in favor of explicit notation through parentheses offers a pathway toward more accessible and intuitive mathematical communication.

However, this research also acknowledges the substantial challenges that any fundamental change to mathematical notation would entail. The investment in current systems is enormous, spanning educational materials, teacher training, technological infrastructure, and international standardization. Professional mathematical communication relies heavily on established conventions, and the transition costs would be considerable across multiple domains. The most critical finding of this analysis is the complete absence of empirical research comparing different approaches to mathematical notation in educational settings. This represents a significant gap in educational research that demands immediate attention. The field has accepted historical conventions without systematic investigation of their educational effectiveness, a situation that contradicts evidence-based approaches to educational policy. Several important gaps in knowledge require further research. First, controlled studies comparing learning outcomes under different notation systems are essential to validate theoretical predictions about cognitive load and learning effectiveness. Second, cross-cultural research examining the relationship between reading patterns and mathematical processing could illuminate whether the advantages of left-to-right calculation apply universally or are culturally specific. Third, longitudinal studies tracking the impact of different notation systems on advanced mathematical development are needed to address concerns about compatibility with higher mathematics.

Future research should also investigate individual differences in response to alternative notation systems, examining whether certain student characteristics predict success with different approaches. Implementation research addressing practical barriers, technology integration, teacher training requirements, and community acceptance would provide crucial guidance for potential educational reforms. The broader significance of this research extends to fundamental questions about educational innovation and evidence-based practice. It demonstrates the importance of questioning established conventions and subjecting them to empirical scrutiny. The findings suggest that many educational practices may persist not because of their effectiveness but because of institutional inertia and historical precedent. This investigation calls for a paradigm shift toward evidence-based evaluation of mathematical conventions, recognizing that notational choices represent human decisions that can and should be optimized for contemporary educational goals. The potential to reduce mathematical anxiety, improve learning outcomes, and increase accessibility represents an opportunity that the educational community cannot afford to ignore. While implementation of alternative approaches would require substantial coordination and investment, the potential benefits for student learning may justify these costs. The research agenda proposed here offers a pathway for systematic investigation that respects existing investments while exploring possibilities for improvement. Ultimately, this research challenges the mathematics education community to move beyond acceptance of historical conventions toward active investigation of optimal approaches to mathematical learning. The question is not whether change is possible, but whether the educational community has the commitment to pursue evidence-based improvement in service of student success. The stakes are too high, and the potential benefits too significant, to allow mathematical conventions to remain unexamined simply because they are established.

Abbreviations

BODMAS	Brackets, Orders, Division and Multiplication, Addition and Subtraction
PEMDAS	Parentheses, Exponents, Multiplication and Division, Addition and Subtraction
SNARC	Spatial-Numerical Association of Response Codes

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