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Research Article

ON THE HOMOMORPHISM OF FINITE GROUPS

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Abstract

In this work we consider the finite groups A and B such that B is abelian and we show that the group $Ho(A, B)$ of homomorphisms from group A to group B is isomorphic to group$B$. If moreover group A is abelian, we obtain isomorphism criteria. We also considered the isomorphism Hom $(A, A) \cong A$for finite abelian groups A.

Keywords: Abelian group, Finite group, Homomorphism, Isomorphism.

INTRODUCTION

The modern theory of abelian groups constitutes a deep part of algebra and is very much investigated nowadays. Particular attention is paid to the groups of homomorphisms of two abelian groups A and Bin the perspective of their complete description which is far from being acquired. Thus, the American mathematician of Hungarian origin Laszlo FUCHS posed in his work [3] his famous problem 30 "describe the group $Ho(A, B)$where A and $B$are abelian groups", a problem open until today even in the case where B is a torsion-free abelian group of rank 1. The investigation of researchers on this problem by considering that the two abelian groups are all finite and more particularly by assuming them to be equal, has given good results in the search for the solution of this problem. In this perspective, we are interested in the groups $Ho(A, B)$ of homomorphisms of two finite (abelian) groups A and B. We established in [10] that the group $Hom(S_3, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$where S_3 is the symmetric group of order 3. In this work, we consider the group $Ho(A, B)$of homomorphisms of two finite groups such that the group B is abelian. For two finite abelian groups A and B, we will define conditions for the group $Hom(A, B) = \{0\}$.

If, moreover, the finite abelian groups A and $B$are cyclic, we will define conditions for the groups $Hom(A, B)$ and$Hom(B, A)$ to be isomorphic to the group B, respectively.

In particular, we will consider the isomorphism $Hom(A, A) \cong A$ for any finite abelian and cyclic group. We will show that the group $Hom(S_3, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

In general, if the group $A = S_3$ is the symmetric group of order 3 and the group $B = \bigoplus_n \mathbb{Z}/2\mathbb{Z}$, $n \in \mathbb{N}$, we show that the group $Hom(A, B) \cong B$.

We will denote by:

- $\mathbb{Z}/n\mathbb{Z}$: the cyclic group of order n;
- A_p : the p-subgroup of the finite group A;
- $Hom(A, B)$: the set of all homomorphisms from group A to group B;
- $P(n)$: the set of prime numbers in the expansion of n;
- $P(A) = \{p \in \mathbb{P}: A_p \neq 0\}$;
- $\sigma(a)$: the order of an element $a \in A$;
- $\bigoplus$: the direct sum
- $|A|$: the cardinality of group A;
- $\mathbb{N}$: the set of natural numbers;
- $\mathbb{N}^*$:the set of non-zero natural numbers;
- $\mathbb{P}$: the set of prime natural numbers;
- Kerφ: the kernel of the homomorphism φ.

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Problems

In works [3], [4] and [10], we posed the following problems:

a) When is $Hom(A, B)$ isomorphic to A or B?
b) Find the abelian groups A such that $Hom(A, A) \cong A$.
c) When is the group $Hom(A, B)$ isomorphic to the group$B$if B is an abelian group and A is any group?

RESULTS

For two finite groups A and B, we describe the following results.

Theorem 3. 1. Let $A$and Bbe two finite abelian groups of order $n$and m, respectively.

The following conditions are equivalent:

1. $Hom(A, B) = \{0\}$
2. $(n, m) = 1$
3. $P(A) \cap P(B) = \emptyset$

Proof. 1⇔2. Indeed, let $A = A_{p_1} \oplus ... \oplus A_{p_k}, \forall p_i \in P(A)$and $B = B_{p_1} \oplus ... \oplus B_{p_t}, \forall p_j \in P(B)$ be the decomposition of the groups $A$and B.

It will follow that

$$Hom(A, B) \cong Hom(A_{p_1} \oplus ... \oplus A_{p_k}, B_{p_1} \oplus ... \oplus B_{p_t}) \cong \bigoplus_{i \in 1,k} \bigoplus_{j \in 1,t} Hom\left(A_{p_i}, B_{p_j}\right).$$

So, $Hom(A,B) = \{0\} \Leftrightarrow \bigoplus_{i \in 1,k} \bigoplus_{j \in 1,t} Hom\left(A_{p_i}, B_{p_j}\right) = \{0\}, \forall p_i \in P(A), \forall p_j \in P(B) \Leftrightarrow Hom\left(A_{p_i}, B_{p_j}\right) = \{0\} \forall p_i, p_j \in P(A) \cap P(B) \Leftrightarrow P(A) \cap P(B) = \emptyset \Leftrightarrow (m,n) = 1$.

Hence the integer n is prime to m.

2⇔3. Noting that $P(A) = P(n)$and$P(B) = P(m)$, we arrive at the required conclusion. Therefore, 1⇔2⇔3. ■

Corollary 3. 1. For every finite abelian group A, there always exists a finite abelian group B such that $Hom(A,B) = \{0\}$.

Proof. Indeed, it suffices to take $B = \mathbb{Z}/p\mathbb{Z}$where$p \in \mathbb{P} - P(A)$. ■

Theorem 3. 2. Let $A$and Bbe finite cyclic abelian groups of order $n$and m, respectively. Then the following conditions are equivalent:

1. $m$divides$n$
2. $Hom(A,B) \cong B$
3. $Hom(B,A) \cong B$.

Proof. 1⇔2. Indeed, for any prime number $p \in P(B)$, we have $|B_p|$divides$|A_p|$ if and only if $m$divides$n$. Therefore, for any prime number $p \in P(B)$, we can write:

$Hom(A_p, B_p) \cong B_p$if and only if $Hom(A,B) \cong B$.
2⇔3. Indeed, for all finite abelian groups $A$and B, by retaining that $Hom(A,B) \cong Hom(B,A)$, we arrive at the required conclusion. ■

Examples 3. 1.

1) $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $B = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, we have: $Hom(A,B) \cong A$.
2) $A = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ and$B = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, we have: $Hom(A,B) = \{0\}$.

Theorem 3. 3. Let A be a finite abelian group. Then the following statements are equivalent:
1. A est cyclique
2. $Hom(A,A) \cong A$.
3. $A_p$is cyclic for all $p \in P(A)$

Proof. Indeed, 1⇔3 because every subgroup of a cyclic group is cyclic.

3⇔2. Then, since the $A_p$are cyclic, it follows:

$Hom(A_p, A_p) \cong Hom(\mathbb{Z}/p^k\mathbb{Z}, \mathbb{Z}/p^k\mathbb{Z}) \cong \mathbb{Z}/p^k\mathbb{Z} \cong A_p$.

Therefore,$Hom(A,A) \cong A$.

2⇔1. Let $Hom(A,A) \cong A$. Since A is a finite abelian group, then $A \cong \oplus A_p$,where$A_p$are cyclic p-groups for all $p \in P(A)$.

But if we have $A_p = \mathbb{Z}/p^k\mathbb{Z} \oplus \mathbb{Z}/p^s\mathbb{Z}$ where $k \neq s$, then we get: $Hom(A_p, A_p) \cong \mathbb{Z}/p^k\mathbb{Z} \oplus \mathbb{Z}/p^s\mathbb{Z} \oplus \mathbb{Z}/p^{\min(k,s)}\mathbb{Z} \oplus \mathbb{Z}/p^{\min(k,s)}\mathbb{Z}$.

So A_p is cyclic for all $p \in P(A)$ and consequently, the group A is also cyclic. ■

Corollary3. 2. Let $A = S_3$be the symmetric group of order 3 and $B = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $Hom(A,B) \cong B$.

Proof. Indeed, let the group $B = \{(0,0),(0,1),(1,0),(1,1)\}$.
Let us first notice that

$A = S_3 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\} \cong Aut(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ with
$\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\alpha_5 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\alpha_6 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and
$\beta_1 = (1\ 2\ 3)$, $\beta_2 = (2\ 1\ 3)$, $\beta_3 = (3\ 2\ 1)$, $\beta_4 = (1\ 3\ 2)$, $\beta_5 = (2\ 3\ 1)$, $\beta_6 = (3\ 1\ 2)$.

Thus, if $\varphi_0 \neq \varphi \in Hom(A,B)$ then $Ker(\varphi) = \{\alpha_1, \alpha_5, \alpha_6\}$ which is a normal subgroup.

Therefore, we obtain:

$$\varphi_1 = \{[(\alpha_1, \alpha_5, \alpha_6),(0,0)],[(\alpha_2, \alpha_3, \alpha_4),(0,1)]\}$$
$$\varphi_2 = \{[(\alpha_1, \alpha_5, \alpha_6),(0,0)],[(\alpha_2, \alpha_3, \alpha_4),(1,1)]\}$$
$$\varphi_3 = \{[(\alpha_1, \alpha_5, \alpha_6),(0,0)],[(\alpha_2, \alpha_3, \alpha_4),(1,0)]\}$$

Ultimately, we have 4 homomorphisms and any non-trivial homomorphism is of order 2.

Hence, we subsequently assert that $Hom(A,B) \cong B$. ■

Theorem 3.4. Let $A = S_3$be the symmetric group of order 3 and $B = \bigoplus_n \mathbb{Z}/2\mathbb{Z}$, $n \in \mathbb{N}$, then $Hom(A,B) \cong B$.

Proof. Indeed, for $n = 0$, it is obvious.

For $n = 1$, see [10] and $n = 2$, see the previous corollary.

Now let $n > 2$. We see that every element in group B is of order 2.

Let $A = S_3 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$such that $\beta_1 = (1\ 2\ 3)$, $\beta_2 = (2\ 1\ 3)$, $\beta_3 = (3\ 2\ 1)$,
$\beta_4 = (1\ 3\ 2)$, $\beta_5 = (2\ 3\ 1)$, $\beta_6 = (3\ 1\ 2)$.

Then if $\varphi_0 \neq \varphi \in Hom(A,B)$and $\varphi(\beta_2) = b \in B$ with $b \neq 0$, it will follow that

$\varphi(\beta_3) = \varphi(\beta_4) = \varphi(\beta_2) = b$because$\beta_3, \beta_4 \in Ker(\varphi)$.

Therefore, $|Hom(A,B)| = |B|$ and any element $\varphi_0 \neq \varphi \in Hom(A,B)$ is of order 2. ■

Conclusion

In this work, we studied the group of homomorphisms of finite groups in general and that of cyclic groups in particular. For two finite abelian groups$A$ and B, we defined conditions for $Hom(A,B) = \{0\}$. Furthermore, if groups A and B are cyclic, we established conditions for the groups $Hom(A,B)$and$Hom(B,A)$ to be isomorphic to group B. In the same vein, we considered the isomorphism $Hom(A,A) \cong A$for any finite cyclic abelian group$A$.

Considering $A = S_3$ and $B = \bigoplus_n \mathbb{Z}/2\mathbb{Z}$, $n \in \mathbb{N}$, we showed that the group $Hom(A, B)$ of homomorphisms from group A to group B is isomorphic to group B.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

Author Contributions: All authors contributed significantly to the writing of this article. They have read and approved the final manuscript.

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