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Research Article

ESSENCE AND PROOF OF ABC CONJECTURE OF INTEGER

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Abstract

A concise proof of abc conjecture of integer and n-term abc conjecture of integer is given in this paper. The essence of abc conjecture of integer is to obtain the number of integer solutions to the abc equation when the abc equation has solutions. Firstly, the combinatorial approximation method to compute the approximate number of integer solutions to linear Diophantine equation with integer subsets as solution set is presented. Secondly, the approximate number of integer solutions to the abc equation was obtained by using combinatorial approximation method, thus providing concise proof of abc conjecture of integer. Finally, the same idea and method is applied to n-term abc conjecture of integer and the results we obtained are consistent with the conjecture.

Keywords: abc conjecture of integer, n-term abc conjecture of integer, Combinatorial approximation method, Linear Diophantine equation, Number of solutions, Set of solutions, Subset of integer.
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INTRODUCTION

The origin and results of the 3-term abc conjecture of integer

The abcconjecture of integer was proposed in the 1980's by J. Oesterle and D.W. Masser[1,2]. This simple statement implies a lot of results and conjectures in number theory.

According to the fundamental theorem of arithmetic, any integer $n \geq 2$ can be written as a product of prime numbers:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} \tag{1}$$

The *radical* (also called *kernel* or *core*)$rad(n)$of$n$is the product of the distinct primes dividing n:

$$rad(n) = p_1 p_2 \cdots p_t \tag{2}$$

abc conjecture of integer: Let $\varepsilon > 0$ be a positive real number. Then there is a constant $C(\varepsilon)$ such that, for any triple a,b,c of coprime positive integers with $a + b = c$, the inequality

$$c \leq C(\varepsilon) rad(abc)^{1+\varepsilon} \tag{3}$$

holds.

Stewart, Tijdeman, and Yu Kunrui obtained some unconditional results of abc conjecture in 1986[3], 1991[4], and 2001[5], and they established the best-known estimate so far:

Theorem (Stewart-Yu Kunrui). There exists an absolute constantκsuch that any abctriple(a, b, c)satisfies

$$\log c \leq \kappa R^{\frac{1}{3}} (\log R)^3 \tag{4}$$

with $R = rad(abc)$.

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Multivariate n-term abc conjecture of integer

In 1994, Browkin and Brzezinski[6] proposed the following multivariate n-term abc conjecture of integer.

Multivariate n-term abc conjecture of integer: Given any integer$n > 2$and any $\varepsilon > 0$, there exists a constant $C_{n,\varepsilon}$. For all integers a_1, a_2,... ,$a_n$and$gcd(a_1, a_2, \cdots, a_n) = 1$,and no proper subsum of $a_1 + a_2 + \cdots + a_n = 0$is equal to 0, we have

$$Max(a_1, a_2, \cdots, a_n) \leq C_{n,\varepsilon} \cdot (rad(a_1 \cdot a_2 \cdots a_n))^{2n-5+\varepsilon} \tag{5}$$

In 2002, Hu Peichu[7] proposed the following multivariate n-term abc conjecture of integer.

Multivariate n-term abc conjecture of integer: Given any integer $n > 2$ and any $\varepsilon > 0$, there exists a constant $C_{n,\varepsilon}$. For all integers a_1, a_2,..., a_n and $gcd(a_1, a_2, \cdots, a_n) = 1$, and no proper sub sum of $a_1 + a_2 + \cdots + a_n = 0$is equal to 0, we have

$$Max(a_1, a_2, \cdots, a_n) \leq C_{n,\varepsilon} \cdot (rad(a_1 \cdot a_2 \cdots a_n))^{1+\varepsilon} \tag{6}$$

When $n = 3$, the proposition degenerates into the abc conjecture of integer.

Results

Because the essence of abcconjecture of integer is to obtain the number of integer solutions to the equation$a + b = c$when the equation $a + b = c$ has solutions, combinatorial approximation methods, and the density transformation method can be employed to determine the number of integer solutions to the Diophantine equation$a + b = c$with countable solution sets. By these methods, the following results can be established.

Theorem 1 Let $\varepsilon > 0$ be a positive real number. Then there is a constant $C(\varepsilon)$ such that, for any triple a,b,c of coprime positive integers with $a + b = c$, the inequality

$$c \leq C(\varepsilon) rad(abc)^{1+\varepsilon}$$

holds.

Theorem 2 Given any integer $n > 2$and any $\varepsilon > 0$, there exists a constant $C_{n,\varepsilon}$. For all integers a_1,a_2, ..., a_n and $gcd(a_1, a_2, \cdots, a_n) = 1$, and no proper sub sum of $a_1 + a_2 + \cdots + a_n = 0$ is equal to 0, we have

$$Max(a_1, a_2, \cdots, a_n) \leq C_{n,\varepsilon} \cdot (rad(a_1 \cdot a_2 \cdots a_n))^{1+\varepsilon}$$

When $n = 3$, the proposition degenerates into the abc conjecture of integer.

To prove theorem 1, theorem 2 and to obtain the number of integer solutions of equation $a + b = c$ and $a_1 + a_2 + \cdots + a_n = 0$, it is necessary to study Lemmas 1-5.

For cases where the solution sets and the number of solution sets of the variables in linear Diophantine equations are either the same or different, we propose the approximate combinatorial method, enumeration method, and density transformation method to compute an approximate number of integer solutions of the linear Diophantine equations whose solution sets are integer subsets or sequences of integer subsets.

Method and Proof of Lemma

Lemma

Why study linear Diophantine equations? More specifically, why analyze the number of integer solutions of linear Diophantine equations whose solution sets consist of sequences of integer subsets?

1. The number of integer solutions to linear Diophantine equations can be determined by combinatorial methods.
2. To find integer solutions to nonlinear or higher-order Diophantine equations is very challenging, whereas counting the number of integer solutions for linear Diophantine equations is comparatively easier.
3. The problem of counting the number of integer solutions to nonlinear or higher-order Diophantine equations can be transformed into the problem of counting the number of integer solutions to linear Diophantine equations whose solution sets are integer subsets.
4. The enumeration method allows for straightforward listing of all integer solutions of linear Diophantine equations when the solution set is an integer sequence, or all of integer solutions are expressed as numerical equations or equalities.
5. Since all integer solutions of linear Diophantine equations with solution sets formed by sequences of integer subsets belong to the integer solutions of linear Diophantine equations with integer sequences as solution sets, the enumeration method likewise facilitates the straightforward listing of all such solutions as numerical equations or equalities.
6. By utilizing the enumeration method, it is straightforward to list all local integer solutions, or all of equations or equalities of local integer solutions of linear Diophantine equations whose solution sets are sequences of integer subsets. Specifically, linear Diophantine equations can explicitly or concisely be represented by the numerical equivalence relations within each local equation ($x_1 + x_2 + \cdots + x_i + \cdots + x_m = i \ (1 \leq i \leq N)$).

This section mainly discusses the number of integer solutions to linear Diophantine equations whose countable solution sets are given by integer sets or integer subsets (sequences). The definitions and notations used herein are consistent with those in reference [8].

Lemma 1 presents the number of integer solutions of linear Diophantine equations with integer set sequence as the solution set. Lemma 2 provides the approximate number of integer solutions of linear Diophantine equations with integer set sequence as the solution set. Lemma 3 provides the approximate number of integer solutions of linear Diophantine equations when the solution sets and the number of the solution sets of variables are either the same or different. Lemma 4 and Lemma 5 present the properties of general linear Diophantine equations when transformed into standard linear Diophantine equations.

Lemma 1 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

The number of integer solutions is well known as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1}$$
$$= \frac{1}{(m-1)!} \frac{N(N-1)(N-2)\cdots\big(N-(m-1)\big)}{N}$$

For any non-negative integer N, the solution set is$x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Lemma 2 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

then the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \frac{1}{(m-1)!}\frac{1}{N}N^m + O(N^{m-2})$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Lemma 3 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ be the number of integer solutionsof the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

then the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) =$$

$\frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2})$

For any non-negative integer N, the solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$, the solution set can be $x_i \neq x_j$, and the number of the solution set is $|x_i| \leq N$, it can be $|x_i| \neq |x_j|$.

Lemma 4 The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Lemma 5 The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof of the Lemma

Lemma 1 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

The number of integer solutions is well known as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1}$$
$$= \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots\big(N-(m-1)\big)}{N}$$

For any non-negative integer$N$, the solution set $x_i = \{1,2,3,\cdots,N\}$, the number of the solution set $|x_i| = N$.

Proof

We consider the number of integer solutions of the linear Diophantine equation where the solution set is an integer set or a sequence of integer set, and a countable solution set. The combinatorial method and the enumeration method are applied.

Combinatorial Method

Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N \tag{7}$$

It is well known that the number of integer solutions is[9]

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N+m-1}{m-1} = C_{N+m-1}^{m-1} \tag{8}$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$. Substitute $x_i - 1 = y_i$ into equation (7) to obtain

$$y_1 + y_2 + \cdots + y_i + \cdots + y_m = N - m \tag{9}$$

The number of integer solutions of equation (9) is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1}(10)$$

The number of integer solutions of equation (9) is equal to the number of integer solutions of equation (7)

$$C_{m+N-1}^{m-1}{=}C_{N-1}^{m-1} \text{ or } \binom{N+m-1}{m-1}{=}\binom{N-1}{m-1} \tag{11}$$

According to combinatorial relations, we have

$$C_N^m = \frac{N}{m}C_{N-1}^{\text{m-1}} \text{ or } C_{N-1}^{m-1} = \frac{m}{N}C_N^m \tag{12}$$

$$C_N^m - C_{N-1}^m = C_{N-1}^{m-1} \tag{13}$$

Equation (13) presents there cursive relation between the m-dimensional linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N)$ and the $(m-1)$-dimensional linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_{m-1} = N)$. This equation (13) also provides the recursive relation between the height$N$linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N)$ and the height $(N-1)$ linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N-1)$. Equation (13) can be used, based on height N and dimension$m$,to prove the number of integer solutions of the linear Diophantine equation by mathematical induction.

Local-Global Principle

The number of integer solutions of equation (7) is denoted as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1} =$$
$$\frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots\big(N-(m-1)\big)}{N} \tag{14}$$

The product $\frac{1}{m!}N(N-1)(N-2)\cdots\big(N-(m-1)\big)$ in the formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$, represents the total number of integer solutions of the equation in m dimensions.

In the product $\frac{1}{m!}\frac{N(N-1)(N-2)\cdots(N-(m-1))}{N}$, the product $\frac{1}{m!}N(N-1)(N-2)\cdots\big(N-(m-1)\big)$ is divided by N, therefore for each N $(N = 1,2,\cdots,i,\cdots,N)$, if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i\ (1 \leq i \leq N))$ haveor not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Linear Diophantine equations can provide clear or concise numerical equivalence relations in each local equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = i\ (1 \leq i \leq N))$.

In linear Diophantine equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$, when $m = 3$, the equation is transformed into $x_1 + x_2 + x_3 = N$.

According to Lemma 5, the number of integer solutions to the equation $x_1 + x_2 + x_3 = N$ is equal to the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$.

According to Lemma 4, the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$ is equal to the number of integer solutions to the equation $x_1 + x_2 - x_3 = 0$ or $x_1 + x_2 = x_3$.

Now, consider the enumeration method to list the integer solutions of linear Diophantine equations with a countable solution set.

From the above, when $m = 3$, the number of integer solutions to the equation $x_1 + x_2 = x_3$or $x_1 + x_2 + x_3 = N$ is equal to $C_{N-1}^{3-1} = C_{N-1}^{2}$.

Table 1 below shows the total number of integer solutions of the 3D equation$x_1 + x_2 = x_3$ where $x_i \leq 10$.

Table 1. Total number of integer solutions of the 3D equation $x_1 + x_2 = x_3$where $x_i \leq 10$

$x_1 + x_2 = x_3$	1	2	3	4	5	6	7	8	9	10
1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10	
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10		
3	1+3=4	2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=10			
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10				
5	1+5=6	2+5=7	3+5=8	4+5=9	5+5=10					
6	1+6=7	2+6=8	3+6=9	4+6=10						
7	1+7=8	2+7=9	3+7=10							
8	1+8=9	2+8=10								
9	1+9=10									
10										
x_2										

$x_1 = \{1,2,3,\cdots,10\}$,$x_2 = \{1,2,3,\cdots,10\}$ and $x_3 = \{1,2,3,\cdots,10\}$

The number of integer solutions to the equation $x_1 + x_2 = x_3$ is equal to

$$C_{N-1}^{3-1} = C_{N-1}^{2} = \frac{(N-1)(N-2)}{2!} \tag{15}$$

When $N \leq 10$, as shown in Table 1.

When $m = 2$, the number of integer solutions to the equation $x_1 + x_2 = N$ is equal to

$$C_N^2 - C_{N-1}^2 = \frac{(N)(N-1)}{2!} - \frac{(N-1)(N-2)}{2!} = C_{N-1}^1 = N - 1 \tag{16}$$

is indicated by the diagonal line in Table 1.

Lemma 1 is proved.

Lemma 2 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ be the number of integer solutions of the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

the approximate number of integer solutions for the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \frac{1}{(m-1)!}\frac{1}{N}N^m + O(N^{m-2})$$

For any non-negative integer N, the solution set is

$x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Proof

Combinatorial approximation method

According to lemma 1, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ has the solution set $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

The number of integer solutions for this equation is $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$.

In the formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$, m is finite and relatively N very small; N is very large and infinite.

The formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$ is

$$\frac{N(N-1)\cdots(N-(m-1))}{m!} - \frac{(N-1)(N-2)\cdots(N-m)}{m!} = \frac{(N-1)(N-2)\cdots(N-(m-1))}{(m-1)!} \tag{17}$$

We make the following approximation $N(N-1)\cdots(N-(m-1)) = N^m + O(N^{m-1})$, $(N-1)(N-2)\cdots(N-m) = \langle N-1\rangle^m + O(N^{m-1})$ and $(N-1)(N-2)\cdots(N-(m-1)) = N^{\langle m-1\rangle} + O(N^{m-2})$.

Obtain

$$\frac{N^m}{m!} - \frac{(N-1)^m}{m!} = \frac{N^{(m-1)}}{(m-1)!} + O(N^{m-2}) \tag{18}$$

Local-Global Principle

The number of solutions to equation (7) is denoted by

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \frac{1}{(m-1)!}\frac{N^m}{N} + O(N^{m-2}) \tag{19}$$

The product $\frac{N^m}{m!}$ in the above formula (18) is the total number of solutions for the m-dimensional equation.

In $\frac{N^m}{m!N}$, the product $\frac{N^m}{m!}$ is divided by N, therefore for each N $(N = 1,2,\cdots,i,\cdots,N)$, if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ $(1 \leq i \leq N)$ have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Equation $x_1 + x_2 = x_3$ has the solution set $x_1 = x_2 = x_3 = \{1,2,3,\cdots,N\}$, and the number of solution set is $|x_1| = |x_2| = |x_3| = N$, and the density of the solution set is $d_1 = d_2 = d_3 = 1$.

Table 2. Total number of solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \leq 10$

$x_1 + x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10		
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10	9+2=11		
3	1+3=4	2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=10	8+3=11	9+3=12		
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10	7+4=11	8+4=12	9+4=13		
5	1+5=6	2+5=7	3+5=8	4+5=9	5+5=10	6+5=11	7+5=12	8+5=13	9+5=14		
6	1+6=7	2+6=8	3+6=9	4+6=10	5+6=11	6+6=12	7+6=13	8+6=14	9+6=15		
7	1+7=8	2+7=9	3+7=10	4+7=11	5+7=12	6+7=13	7+7=14	8+7=15	9+7=16		
8	1+8=9	2+8=10	3+8=11	4+8=12	5+8=13	6+8=14	7+8=15	8+8=16	9+8=17		
9	1+9=10	2+9=11	3+9=12	4+9=13	5+9=14	6+9=15	7+9=16	8+9=17	9+9=18		
10											
x_2											

Table 2 lists the total number of solutions for the 3D equation $x_1 + x_2 = x_3$ satisfies $x_i \leq 10$.

$x_1 = \{1,2,3,\cdots,10\}$, $x_2 = \{1,2,3,\cdots,10\}$ and $x_3 = \{1,2,3,\cdots,10\}$

When $m = 3$, the number of integer solutions of the equation $x_1 + x_2 = x_3$ or $x_1 + x_2 + x_3 = N$ is equal to $NS \sim \frac{N^2}{2!}$.

When $m = 2$, the number of integer solutions of the equation $x_1 + x_2 = N$ is equal to

$NS \sim \frac{N^2}{2!} - \frac{(N-1)^2}{2!} \sim \frac{1}{1!}N^1$. Shown by the diagonal line in Table 2.

The proof of Lemma 2 is completed.

Lemma 3 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ denote the number of integer solutions of the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

and the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2})$$

For any non-negative integer N, the solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$, the solution set can be $x_i \neq x_j$, and the number of the solution sets is $|x_i| \leq N$, and it can be $|x_i| \neq |x_j|$.

Proof

Combinatorial approximation method

According to lemma 2, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ has the solution set $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$, for the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$, has the following approximate number of integer solutions:

$$NS = \frac{N^m}{m!} - \frac{(N-1)^m}{m!} = \frac{1}{(m-1)!}N^{(m-1)} + O(N^{m-2}) \quad (20)$$

We rewrite the above expression as

$$\frac{N^m}{m!} - \frac{(N-1)^m}{m!} = \frac{1}{(m-1)!}\frac{1}{N}N^m + O(N^{m-2}) \quad (21)$$

The solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$ and it can be $x_i \neq x_j$, and the number of the solution set is $|x_i| \leq N$, and it can be $|x_i| \neq |x_j|$, and the density of the solution set is $d_i = \frac{|x_i|}{N}$.

The above expression (21) multiplied by the density $(d_1 \cdot d_2 \cdot \cdots \cdot d_i \cdot \cdots \cdot d_m)$ of the solution sets

$$(d_1 \cdot d_2 \cdot \cdots \cdot d_m)\left\{\frac{N^m}{m!} - \frac{(N-1)^m}{m!}\right\} \sim \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot d_2 \cdot \cdots \cdot d_m)N^m \quad (22)$$

$$\text{Left side} = \frac{(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)}{m!} - \frac{(d_1 \cdot N - d_1)(d_2 \cdot N - d_2)\cdots(d_m \cdot N - d_m)}{m!} \quad (23)$$

Since $0 < d_i \leq 1$, we make the following approximation $(d_1 \cdot N - d_1)(d_2 \cdot N - d_2)\cdots(d_m \cdot N - d_m) \sim (d_1 \cdot N - 1)(d_2 \cdot N - 1)\cdots(d_m \cdot N - 1)$.

The above formula can be approximated as

$$\text{Left side} = \frac{(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)}{m!} - \frac{(d_1 \cdot N - 1)(d_2 \cdot N - 1)\cdots(d_m \cdot N - 1)}{m!}$$

$$\text{Left side} = \frac{|x_1||x_2|\cdots|x_m|}{m!} - \frac{(|x_1|-1)(|x_2|-1)\cdots(|x_m|-1)}{m!}$$

$$\text{Right side} = \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot d_2 \cdot \cdots \cdot d_m)N^m = \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)$$

$$\text{Right side} = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m|$$

Left side = Right side

$$\frac{|x_1||x_2|\cdots|x_m|}{m!} - \frac{(|x_1|-1)(|x_2|-1)\cdots(|x_m|-1)}{m!} = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2}) \quad (24)$$

Consider the non-uniform distribution of the number of integer solutions caused by the combination of density d_i in each m dimension with N change, and then $\frac{1}{(m-1)!}$ is modified to $c(m, d_i)$. $c(m, d_i)$ can be obtained through calculation based on precision; that is

$$N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| = c(m, d_i)\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2}) \quad (25)$$

Local-Global Principle

The number of integer solutions of equation (7) is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2}) \quad (26)$$

The product $\frac{|x_1||x_2|\cdots|x_m|}{m!}$ in the above formula (24) is the total number of integer solutions in dimension m.

In $\frac{|x_1||x_2|\cdots|x_m|}{Nm!}$, the product $\frac{|x_1||x_2|\cdots|x_m|}{m!}$ is divided by N, N ($N = 1,2,\cdots,i,\cdots,N$), if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ $(1 \le i \le N)$ have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Example 1. Prime number

Solution set $p = \{2, 3, 5, \cdots, p_i, \cdots, N\}$, $p_i \le N$, the number of prime solution sets is $|p| = \frac{N}{\ln N}$, and the density of the prime solution sets is $d_1 = d_2 = \frac{1}{\ln N}$. The approximate number of integer solutions of theequation $p_1 + p_2 = N$ is $NS \sim c\frac{|p_1|\cdot|p_2|}{1!N} \sim \frac{0.62N}{(\ln N)^2}$ (including duplicate solutions). For details on Goldbach's conjecture and the number of integer solutions of Goldbach's equation $p_1 + p_2 = N$, please see reference [8].

Table 3. Total number of integer solutions of 2D equation $p_1 + p_2 = N$ where $N \le 10$

$p_1 + p_2 = N$	2	3	5	7	p_1
2	2+2=4				
3		3+3=6	5+3=8	7+3=10	
5		3+5=8	5+5=10	7+5=12	
7		3+7=10	5+7=12	7+7=14	
p_2					

$x_i = \{2, 3, 5, \cdots, p_i, \cdots, N\} \subset \{1,2,3,\cdots,N\}$,$|x_i| \le N = 10$

Example 2. Residue class

In the equation $a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_mx_m = N$, the solution set of the residue class is $a_ix_i = \{a_i \cdot 1, a_i \cdot 2, \dots, a_i \cdot k, \dots, N\}$.

By substitution $a_ix_i = y_i$,

we can obtain $y_1 + y_2 + \cdots + y_i + \cdots + y_m = N$.

Solution set y_i is $a_ix_i = \{a_i \cdot 1, a_i \cdot 2, \dots, a_i \cdot k, \dots, N\}$, therefore $a_i \cdot k \le N$, and the number of the solution set is $|a_ix_i| = \frac{N}{a_i}$.

In the equation $x_1 + 2x_2 = x_3$, the solution sets x_1,$2x_2$ and x_3 are

$$x_1 = x_3 = \{1,2,3,\cdots,k,\cdots,N\}$$
$$2x_2 = \{2, 4, \cdots, 2k, \cdots, N\}$$

The number of the solution set $|x_1| = |x_3| = N$, $|2x_2| = \frac{N}{2}$.

The number of integer solutions to the equation $x_1 + 2x_2 = x_3$ is

$$NS \sim \frac{|x_1|\cdot|2x_2|\cdot|x_3|}{2!N} = \frac{N\cdot\frac{N}{2}\cdot N}{2N} = \frac{N^2}{4}.$$

$$x_i \subseteq \{1,2,3,\cdots,N\},\ x_i \ne x_j, |x_i| \ne |x_j|, |x_i| \le N = 10$$

Density Transformation Method

We now discuss the density transformation of linear Diophantine equations whose solution set is an integer subset.

Let a_1,a_2,...,a_m be positive integers to satisfy $gcd(a_1, a_2, \cdots, a_m) = 1$. In addition, let $NS = N(a_1, a_2, \cdots, a_m; N)$ be the number of integer solutions of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_mx_m = N \quad (27)$$

For non-negative integers x_1,x_2,...,x_m. According to Schur's theorem, the approximate number of integer solutions of equation (27) is

$$NS = N(a_1, a_2, \cdots, a_m; N) = \frac{N^{m-1}}{(m-1)!\cdot a_1\cdot a_2\cdot\cdots\cdot a_m} + O(N^{m-2}) \quad (28)$$

For non-negative integers$N$, provided that$N$is sufficiently large, especially when there exists an integerN_1such that each$N \ge N_1$has at least one representation [9].

Equation $$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N \quad (29)$$

Non-negative integers x_1,x_2,...,x_m, and the solution sets are integer subsets.

$x_i = \{(a_i)_1, (a_i)_2, (a_i)_3, \cdots, N\} \subseteq \{1,2,3,\cdots,N\}(1 \le i \le m)$, and can be $x_i \ne x_j$. When $(a_i)_k \le N$, and the density of the solution set is $d_i = \frac{|x_i|}{N}$, and the number of the solution set is

$|x_i| = d_i \cdot N \le N$.

Transform $x_i = \frac{1}{d_i} \cdot y_i$, the solution set is $y_i = \{1,2,3,\cdots,N\}(1 \le i \le m)$is the integer set.

Solution set $\frac{1}{d_i} \cdot y_i = \left\{\frac{1}{d_i} \cdot 1, \frac{1}{d_i} \cdot 2, \dots, \frac{1}{d_i} \cdot k, \dots, N\right\} \subseteq \{1, 2, 3, \cdots, k, \cdots, N\}$, and the number of the solution set is $\left|\frac{1}{d_i} \cdot y_i\right| = d_i \cdot N \le N$, the solution set is $\frac{1}{d_i} \cdot y_i$, the density is d_i, and there are

Table 4. Total number of integer solutions of the equation $x_1 + 2x_2 = x_3$, where$x_{1,\ 3} \le 10, 2x_2 \le 10$

$x_1 + 2x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10	9+2=11		
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10	7+4=11	8+4=12	9+4=13		
6	1+6=7	2+6=8	3+6=9	4+6=10	5+6=11	6+6=12	7+6=13	8+6=14	9+6=15		
8	1+8=9	2+8=10	3+8=11	4+8=12	5+8=13	6+8=14	7+8=15	8+8=16	9+8=17		
10											
$2x_2$											

$$\frac{1}{d_1}\cdot y_1+\frac{1}{d_2}\cdot y_2+\cdots+\frac{1}{d_i}\cdot y_i+\cdots+\frac{1}{d_m}\cdot y_m=N \quad (30)$$

According to formula (28), the approximate number of integer solutions of equation (30) is

$$NS=N\left(\frac{1}{d_1},\frac{1}{d_2},\cdots,\frac{1}{d_m};N\right)=\frac{m}{N}\frac{1}{(m)!}(d_1\cdot N)(d_2\cdot N)\cdot\ldots\cdot(d_m\cdot N)+O(N^{m-2}) \quad (31)$$

where $d_i\cdot N=|x_i|$. The approximate number of integer solutions of equation (30) is also

$$NS=\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}+O(N^{m-2}) \quad (32)$$

Lemma 3 is proved.

For further discussion on the cases of unequal density $d_i\cdot N=|x_i|$ and applications, see reference [8, 10, 11].

Lemma 4 The number of integer solutions of the equation $x_1\pm x_2\pm\cdots\pm x_i\pm\cdots\pm x_m=N$ is equal to the number of integer solutions of the equation $x_1+x_2+\cdots+x_i+\cdots+x_m=N$.

Proof

Combinatorial approximation method

In the equation $x_1\pm x_2\pm\cdots\pm x_i\pm\cdots\pm x_m=N$, (33)

By substituting $x_1=y_1$, $\pm x_i=y_i(2\le i\le m)$

we can obtain $y_1+y_2+\cdots+y_i+\cdots+y_m=N$. (34)

When x_i, $y_i(1\le i\le m)\subset\{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i\subset\{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N\le -k\le -1$, the number of the solution set is $|x_1|=|y_1|$, $|-x_i|=|x_i|=|y_i|(2\le i\le m)$.

According to lemma 3, the approximate number of integer solutions of equation (33) can be expressed as

$$NS\sim c\frac{|x_1|\cdot|\pm x_2|\cdots|\pm x_m|}{N\cdot(m-1)!}\sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}+O(N^{m-2}) \quad (35)$$

According to lemma 3, the approximate number of integer solutions of equation (34) can be expressed as

$$NS\sim c\frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!}\sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}+O(N^{m-2}) \quad (36)$$

When x_i, $y_i(1\le i\le m)=\{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i=\{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N\le -k\le -1$, the number of the solution set is $|x_1|=|y_1|=N$, $|-x_i|=|x_i|=|y_i|=N(2\le i\le m)$.

According to lemma 2, the approximate number of integer solutions of equation (33) can be expressed as

$$NS\sim c\frac{|x_1|\cdot|\pm x_2|\cdots|\pm x_m|}{N\cdot(m-1)!}\sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}\sim c\frac{N^{m-1}}{(m-1)!}+O(N^{m-2}) \quad (37)$$

According to lemma 2, the approximate number of integer solutions of equation (34) can be expressed as

$$NS\sim c\frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!}\sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}\sim c\frac{N^{m-1}}{(m-1)!}+O(N^{m-2}) \quad (38)$$

The number of integer solutions of equation (33) is equal to the number of integer solutions of equation (34).

Proof of Lemma 4 is completed.

Lemma 5 The number of integer solutions of equation $x_1\pm x_2\pm\cdots\pm x_i\pm\cdots\pm x_m=0$ is equal to the number of integer solutions of equation $x_1+x_2+\cdots+x_i+\cdots+x_m=N$.

Proof

Combinatorial approximation method

Equation $x_1\pm x_2\pm\cdots\pm x_i\pm\cdots\pm x_m=0$ (39)

becomes $x_1\pm x_2\pm\cdots\pm x_i\pm\cdots\pm x_m+N=N$ (40)

By substitution $x_1+N=y_1$ and $\pm x_i=y_i(2\le i\le m)$

we can obtain $y_1+y_2+\cdots+y_i+\cdots+y_m=N$. (41)

Since N is a constant; therefore, it is not included the number of the solution set. The number of solution set $|x_1+N|$ is only regarded as $|x_1+N|=|x_1|=|y_1|$.

When x_i, $y_i(1\le i\le m)\subset\{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i=\{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N\le -k\le -1$, the number of the solution set is $|-x_i|=|x_i|=|y_i|(2\le i\le m)$.

According to lemma 3, the approximate number of integer solutions of equation (39) can be expressed as

$$NS\sim c\frac{|x_1+N|\cdot|\pm x_2|\cdots|\pm x_m|}{N\cdot(m-1)!}\sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}+O(N^{m-2}) \quad (42)$$

According to lemma 3, the approximate number of integer solutions of equation (41) can be expressed as

$$NS\sim c\frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!}\sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}+O(N^{m-2}) \quad (43)$$

Since N is a constant; therefore, it is not included the number of the solution set. The number of solution set $|x_1+N|$ is only regarded as $|x_1+N|=|x_1|=|y_1|=N$.

When x_i, $y_i(1\le i\le m)=\{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i=\{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N\le -k\le -1$, the number of the solution set is $|-x_i|=|x_i|=|y_i|=N(2\le i\le m)$.

According to lemma 2, the approximate number of integer solutions of equation (39) can be expressed as

$$NS\sim c\frac{|x_1+N|\cdot|\pm x_2|\cdots|\pm x_m|}{N\cdot(m-1)!}\sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}\sim c\frac{N^{m-1}}{(m-1)!}+O(N^{m-2}) \quad (44)$$

According to lemma 2, the approximate number of integer solutions of equation (41) can be expressed as

$$NS\sim c\frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!}\sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!}\sim c\frac{N^{m-1}}{(m-1)!}+O(N^{m-2}) \quad (45)$$

The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof of Lemma 5 is completed.

Enumeration Method of Lemma 4 and Lemma 5

Table 5. Total number of integer solutions of the 3D equation $x_1 - x_2 = x_3$ where $x_i \leq 10$

$x_1 - x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1-1=0	2-1=1	3-1=2	4-1=3	5-1=4	6-1=5	7-1=6	8-1=7	9-1=8		
2		2-2=0	3-2=1	4-2=2	5-2=3	6-2=4	7-2=5	8-2=6	9-2=7		
3			3-3=0	4-3=1	5-3=2	6-3=3	7-3=4	8-3=5	9-3=6		
4				4-4=0	5-4=1	6-4=2	7-4=3	8-4=4	9-4=5		
5					5-5=0	6-5=1	7-5=2	8-5=3	9-5=4		
6						6-6=0	7-6=1	8-6=2	9-6=3		
7							7-7=0	8-7=1	9-7=2		
8								8-8=0	9-8=1		
9									9-9=0		
10											
x_2											

For globe

for $x_1 - x_2 = N$ $(N \geq k \geq 0)$, see total number in Table 5.
for $x_1 + x_2 = N$ $(0 \leq k \leq N)$, see total number in Table 1.

For local

For $x_1 - x_2 = k$, see diagonal at k in Table 5.
For $x_1 + x_2 = N - k$, see diagonal at $N - k$ in Table 1.
For other applications and validations of the approximate combinatorial method, see reference [8, 10, 11].

Proof of Theorem 1 and Theorem 2

The proof of abc conjecture of integer

Theorem 1 Let $\varepsilon > 0$ be a positive real number. Then there is a constant $C(\varepsilon)$ such that, for any triple a,b,c of coprime positive integers with $a + b = c$, the inequality

$$c \leq C(\varepsilon) rad(abc)^{1+\varepsilon}$$

holds.

Proof

RecallLemma3. When $m = 3$, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ change into $x_1 + x_2 + x_3 = N$. The approximate number of integer solutions to the equation$x_1 + x_2 + x_3 = N$ is

$$NS = \frac{|x_1| \cdot |x_2| \cdot |x_3|}{2!N} + O(N) \quad (46)$$

In the abc conjecture of integer, satisfies$a + b = c$, that is, the equation$a + b = c$has solutions, and the number of integer solutions of the equation$a + b = c$is more than or equal to 1.

According to lemma 4, the number of integer solutions of the equation $a + b = c$ is equal tothe number of integer solutions of the equation $a + b + c = 0$.

According to lemma 5, the number of integer solutions of equation $a + b + c = 0$ is equal to the number of integer solutions of the equation $a + b + c = N$, where $N \geq Max(a, b, c)$is an integer constant.

Transform the equation$a + b = c$ into

$$a + b + c = N. \quad (47)$$

In order to compute the number of integer solutions of the equation (47), in the equation $x_1 + x_2 + x_3 = N$, when $x_1 = a, x_2 = b$ and $x_3 = c$, the approximate number of integer solutions to the equation $a + b + c = N$is

$$NS \sim \frac{|x_1| \cdot |x_2| \cdot |x_3|}{2!N} = \frac{|a| \cdot |b| \cdot |c|}{2!c} + O(c) \quad (48)$$

Let's analyze the solution set and the number of solution set for$a$,$b$and$c$from the following two situations.

1) When the solution set a,b and c are the set of integers, that is, the integer$a$,$b$and$c$are not decomposed into the product of prime numbers.

The set $a, b, or, c = \{1, 2, 3, \cdots, j, \cdots, N\}$is the set of all integer.

When $a \leq N$,$b \leq N$and$c \leq N$, the number of the solution set is $|a|$=N,$|b|$=N and $|c|$=N. We have the approximate number of integer solutions to the equation$a + b + c = N$

$$NS \sim \frac{1}{2\ !} \frac{|a| \times |b| \times |c|}{N} = \frac{1}{2\ !} \frac{N \times N \times N}{N} = \frac{rad(a) \cdot rad(b) \cdot rad(c)}{2! \cdot N} = \frac{rad(abc)}{2! \cdot c} = \frac{k \cdot rad(abc)}{c} \geq 1 \quad (49)$$

Else $$\frac{k \cdot \left(rad(abc)\right)^{1+\varepsilon}}{c} \geq \frac{k \cdot rad(abc)}{c} \geq 1 \quad (50)$$

or $$k \cdot \left(rad(abc)\right)^{1+\varepsilon} \geq c \quad (51)$$

2) When the solution sets$a$,$b$and$c$are decomposed into the product form of primes, the solution sets a,b and c are following form.

$$a, b, or, c = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} \quad (52)$$

The number $|a|$,$|b|$ and $|c|$ of solution sets a,b and c is$|a|$,$|b|$ or $|c|$=$rad(a, b, or, c) = p_1 p_2 \cdots p_t$.

Let's look at the following example $a, b, or, c \leq N$.

For example, $a = 2^3 \leq 8$, the number of solution sets is $|a| = rad(2^3) = 2$, and the solution set is $a = \{1, \quad 2\}$.

For example, $a = 3^2 \leq 9$, the number of solution sets is $|a| = rad(3^2) = 3$, and the solution set is$a = \{1, \quad 2, \quad 3\}$.

For example, $a = 2^3 3^2 \leq 72$, the number of solution set is$|a| = rad(2^3 \cdot 3^2) = 2 \cdot 3 = 6$, and the solution set is$a = \{1 \cdot 1, \quad 1 \cdot 2, \quad 1 \cdot 3; \quad 2 \cdot 1, \quad 2 \cdot 2, \quad 2 \cdot 3\}$.

The function $rad(n)$ has two functions. First, the power of the function $rad(a, b, or, c) = p_1 p_2 \cdots p_t$ does not affect the number of solution set. Second, the function $rad(n)$ is a multiplicative function that acts as a combinatorial product.

The function $rad(a, b, or, c) = p_1 p_2 \cdots p_t$ is the number of solution set.

The set $\{p_1 p_2 \cdots p_t | p_1 p_2 \cdots p_t = rad(a, b, or, c)\}$ is a subset of all integers $\{1,2, \cdots, N\}$.

$$\{p_1 p_2 \cdots p_t | p_1 p_2 \cdots p_t = rad(a, b, or, c)\} \subseteq \{p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} | p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} = a, b, or, c\} \subseteq \{1,2, \cdots, N\}$$

When $a \leq N$,$b \leq N$ and $c \leq N$, the number of solution set is$|a|$=$rad(a) = p_1 p_2 \cdots p_t$, $|b|$=$rad(b) = p_1 p_2 \cdots p_t$, and $|c|$=$rad(c) = p_1 p_2 \cdots p_t$.

The equation $a + b = c$ in the abc conjecture has solutions. We have the approximate number of solutions to equation $a + b + c = N$

$$NS \sim \frac{1}{2\,!} \frac{|a| \times |b| \times |c|}{N} = \frac{rad(a) \cdot rad(b) \cdot rad(c)}{2! \cdot N} = \frac{rad(abc)}{2! \cdot c} = \frac{k \cdot rad(abc)}{c} \geq 1 \quad (53)$$

Else $$\frac{k \cdot (rad(abc))^{1+\varepsilon}}{c} \geq \frac{k \cdot rad(abc)}{c} \geq 1 \quad (54)$$

Or $$k \cdot (rad(abc))^{1+\varepsilon} \geq c \quad (55)$$

Where $k = \frac{1}{2\,!}$. The denominator c can be understood as height.

The quantity$\varepsilon$is an adjustment to the approximate solutionNSin above.

Proof is completed.

Similarly, multivariate n-term abc conjecture of integer can be proven.

The proof of n-term abc conjecture of integer

Theorem 2 Given any integer$n > 2$and any $\varepsilon > 0$, there exists a constant $C_{n,\varepsilon}$. For all integers a_1,a_2, ..., a_n and $gcd(a_1, a_2, \cdots, a_n) = 1$, and no proper subsum of $a_1 + a_2 + \cdots + a_n = 0$ is equal to 0, we have

$$Max(a_1, a_2, \cdots, a_n) \leq C_{n,\varepsilon} \cdot (rad(a_1 \cdot a_2 \cdots a_n))^{1+\varepsilon}$$

When $n = 3$, the proposition degenerates into the abc conjecture of integer.

Proof

In the n-term abc conjecture of integer, satisfies $a_1 + a_2 + \cdots + a_n = 0$, that is, the equation $a_1 + a_2 + \cdots + a_n = 0$ has solutions, and the number of solutions to the equation $a_1 + a_2 + \cdots + a_n = 0$ is more than or equal to 1.

According to the transformation of the equation in lemma 4 and lemma 5, we only need to compute the number of solutions to the equation

$$a_1 + a_2 + \cdots + a_n = N \quad (56)$$

Let's analyze the solution set and the number of solution sets for a_i from the following two situations.

1) When the solution set a_i is the set of integers, that is, the integer$a_i$is not decomposed into the product of prime numbers.

$$a_i = \{1, 2, \quad 3, \cdots, j, \cdots, N\}$$

When $j \leq N$, the number of solution set is $|a_i|$=N. We have an approximate number of integer solutions of the equation $a_1 + a_2 + \cdots + a_n = N$,

where $N = Max(|a_1|, |a_2|, \cdots, \quad |a_n|)$,

$$NS \sim \frac{1}{(n-1)!} \frac{|a_1| \times |a_2| \times \cdots \times |a_n|}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} = \frac{1}{(n-1)!} \frac{rad(a_1) \cdot rad(a_2) \cdot \cdots \cdot rad(a_n)}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} = \frac{1}{(n-1)!} \frac{rad(a_1 a_2 \cdots a_n)}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} = \frac{k \cdot rad(a_1 a_2 \cdots a_n)}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} \geq 1 \quad (57)$$

Else $$\frac{k \cdot (rad(a_1 a_2 \cdots a_n))^{1+\varepsilon}}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} \geq \frac{k \cdot rad(a_1 a_2 \cdots a_n)}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} \geq 1 \quad (58)$$

or $k \cdot (rad(a_1 a_2 \cdots a_n))^{1+\varepsilon} \geq Max(|a_1|, |a_2|, \cdots, \quad |a_n|)$(59)

2) When the solution sets$a_i$are decomposed into the product form of primes, the solution sets a_i are following form.

$$a_i = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} \quad (60)$$

The number of solution sets$a_i$is $|a_i|$=$rad(a_i) = p_1 p_2 \cdots p_t$.

We have the approximate number of integer solutions of equation $a_1 + a_2 + \cdots + a_n = N$,

where $N = Max(|a_1|, |a_2|, \cdots, \quad |a_n|)$,

$$NS \sim \frac{1}{(n-1)!} \frac{|a_1| \times |a_2| \times \cdots \times |a_n|}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} = \frac{1}{(n-1)!} \frac{rad(a_1) \cdot rad(a_2) \cdot \cdots \cdot rad(a_n)}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} = \frac{1}{(n-1)!} \frac{rad(a_1 a_2 \cdots a_n)}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} = \frac{k \cdot rad(a_1 a_2 \cdots a_n)}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} \geq 1 \quad (61)$$

Else $$\frac{k \cdot (rad(a_1 a_2 \cdots a_n))^{1+\varepsilon}}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} \geq \frac{k \cdot rad(a_1 a_2 \cdots a_n)}{Max(|a_1|, |a_2|, \cdots, \ |a_n|)} \geq 1 \quad (62)$$

Or$k \cdot (rad(a_1 a_2 \cdots a_n))^{1+\varepsilon} \geq Max(|a_1|, |a_2|, \cdots, \quad |a_n|)$ (63)

Where $k = \frac{1}{(n-1)!}$, and the denominator $Max(|a_1|, |a_2|, \cdots, \quad |a_n|)$can be understood as height.

Proof is completed.

From above discussion, we can see that the larger the number of dimensions$m$ or n, the greater the number of solutions of the equation.

Nevanlinna theory deals with the growth of zeros and poles (roots or solution sets) of meromorphic functions. According to Vojta's conjecture, there is a corresponding (similar) relationship between Nevanlinna theory in function fields and the abc conjecture in number fields (including integers). Since the general Fermat conjecture can be derived from the abc conjecture and also concerns the number of zeros (roots or solutions) of equations, Nevanlinna theory, the abc conjecture, Vojta's conjecture, and the general Fermat conjecture all those problems are the same class of problems related to the number of zeros (roots or solutions) of equations.

This article presents an approximate combinatorial method for estimating the number of integer solutions to general Diophantine equations with countable solution sets. The applicable countable solution sets include integers, rational numbers, prime numbers, algebraic numbers, polynomials, algebraic number fields, and function fields. Some problems can also be transformed into counting number of solutions of equations. For applications of this method in other areas, such as the essence and number of solutions of the abc conjecture for algebraic numbers, polynomials, algebraic number fields, and function fields, the number of solutions to the general Fermat conjecture, and so forth, please refer to the reference [8].

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