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Research Article

ON ERDOS AND TURAN CONJECTURE ON ARITHMETIC PROGRESSIONS

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Abstract

A concise proof of Erdos and Turan conjecture on arithmetic progressions is given in this paper. Whether the set $A \subset \{1,2,\cdots,N\}$ $(N \geq 2)$ contains no non-trivial arithmetic progressions is equivalent to whether the corresponding linear Diophantine equation has no integer solutions, so the combinatorial approximation method to obtain the approximate number of integer solutions to the linear Diophantine equation with integer subsets as solution set is introduced. In different situations (Bloom and Sisask's situation, Behrend's situation, the case of powers of integer), we assume different forms or models of density of the progressions or the integer subsets to compute the approximate number of integer solutions to the corresponding linear Diophantine equation. When the corresponding linear Diophantine equation has no integer solutions, we can get the density of progressions. The approximate number of integer solutions of corresponding linear Diophantine equations of the 2-term and 3-term arithmetic progressions and the density of the 2-term and 3-term arithmetic progressions are computed to validate the combinatorial approximation method. The results of 2-term and 3-term arithmetic progressions are improved and the results of n-term arithmetic progressions are obtained.

Keywords: Erdos and Turan conjecture on arithmetic progressions, Arithmetic progressions, Linear Diophantine equation, Combinatorial approximation method, Number of solutions, Set of solutions, Progressions of integer subsetsr.

INTRODUCTION

Erdos and Turan conjecture

A well-known conjecture of Erdos and Turan states that if$A$is a set of positive integers such that $\sum_{n\in A}\frac{1}{n}$diverges then$A$contains arbitrarily long arithmetic progressions[1].

Erdos and Turan conjecture. Suppose that $A = \{a_1 < a_2 < \cdots\}$ is an infinite sequence of integers such that $\sum \frac{1}{a_i} = \infty$ then A contains arbitrarily long arithmetic progressions [1].

Because $A \subset \{1,2,\cdots,N\}$ $(N \geq 2)$ is a set with no non-trivial three-term arithmetic progressions, *i.e.* no solutions to $x + y = 2z$ with $x \neq y$. In the discussion on Erdos and Turan conjecture on arithmetic progressions, whether the set $A \subset \{1,2,\cdots,N\}$ $(N \geq 2)$ contains no non-trivial arithmetic progressions is equivalent to whether the corresponding linear Diophantine equation$x + y = 2z$ has no integer solutions. For determining the number of integer solutions of the linear Diophantine equation, first, we introduce the combinatorial method, approximate combinatorial method, and enumeration method to compute the number of integer solutions of linear Diophantine equations whose solution set is integer set (integer sequence) and is countable. Second, for cases where the solution sets and the number of solution sets of the variables in linear Diophantine equations are either the same or different, we propose the approximate combinatorial method, enumeration method, and density transformation method to compute an approximate number of integer solutions of the linear Diophantine equations whose solution sets are integer subsets or sequences of integer subsets.

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METHOD AND PROOF OF LEMMA

Lemma

Why study linear Diophantine equations? More specifically, why analyze the number of integer solutions of linear Diophantine equations whose solution sets consist of sequences of integer subsets?

1. The number of integer solutions to linear Diophantine equations can be determined by combinatorial methods.
2. To find integer solutions to nonlinear or higher-order Diophantine equations is very challenging, whereas counting the number of integer solutions for linear Diophantine equations is comparatively easier.
3. The problem of counting the number of integer solutions to nonlinear or higher-order Diophantine equations can be transformed into the problem of counting the number of integer solutions to linear Diophantine equations whose solution sets are integer subsets.
4. The enumeration method allows for straightforward listing of all integer solutions of linear Diophantine equations when the solution set is an integer sequence, or all of integer solutions are expressed as numerical equations or equalities.
5. Since allinteger solutions of linear Diophantine equations with solution sets formed by sequences of integer subsets belong to the integer solutions of linear Diophantine equations with integer sequences as solution sets, the enumeration method likewise facilitates the straightforward listing of all of such solutions as numerical equations or equalities.
6. By utilizing the enumeration method, it is straightforward to list all local integer solutions, or all of equations or equalities of local integer solutions of linear Diophantine equations whose solution sets are sequences of integer subsets. Specifically, linear Diophantine equations can

explicitly or concisely be represented by the numerical equivalence relations within each local equation ($x_1 + x_2 + \cdots + x_i + \cdots + x_m = i\ (1 \le i \le N)$).

This section mainly discusses the number of integer solutions to linear Diophantine equations whose countable solution sets are given by integer sets or integer subset (sequences). The definitions and notations used herein are consistent with those in reference [8].

Lemma 1 presents the number of integer solutions of linear Diophantine equations with integer set sequence as the solution set. Lemma 2 provides the approximate number of integer solutions of linear Diophantine equations with integer set sequence as the solution set. Lemma 3 provides the approximate number of integer solutions of linear Diophantine equations when the solution sets and the number of the solution sets of variables are either the same or different. Lemma 4 and Lemma 5 present the properties of general linear Diophantine equations when transformed into standard linear Diophantine equations.

Lemma 1 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

The number of integer solutions is well known as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1}$$
$$= \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots\left(N-(m-1)\right)}{N}$$

For any non-negative integer N, the solution set is$x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Lemma2 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

then the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) \sim \frac{1}{(m-1)!}\frac{1}{N}N^m + O(N^{m-2})$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Lemma 3 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

then the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$$
$$\sim \frac{1}{(m-1)\ !}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2})$$

For any non-negative integer N, the solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$, the solution set can be $x_i \ne x_j$, and the number of the solution set is $|x_i| \le N$, it can be $|x_i| \ne |x_j|$.

Lemma 4 The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Lemma 5 The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof of Lemma

Lemma 1 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

The number of integer solutions is well known as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1}$$
$$= \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots\left(N-(m-1)\right)}{N}$$

For any non-negative integer N, the solution set $x_i = \{1,2,3,\cdots,N\}$, the number of the solution set $|x_i| = N$.

Proof

We consider the number of integer solutions of the linear Diophantine equation where the solution set is an integer set or a sequence of integer set, and a countable solution set. The combinatorial method and the enumeration method are applied.

Combinatorial Method

Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the linear Diophantine equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ (1)

It is well known that the number of integer solutions is[2]

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N+m-1}{m-1} = C_{N+m-1}^{m-1} \quad (2)$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$. Substitute $x_i - 1 = y_i$into equation (1) to obtain

$$y_1 + y_2 + \cdots + y_i + \cdots + y_m = N - m \quad (3)$$

The number of integer solutions of equation (3) is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1} \quad (4)$$

The number of integer solutions of equation (3) is equal to the number of integer solutions of equation (1)

$$C_{m+N-1}^{m-1}=C_{N-1}^{m-1} \text{ or } \binom{N+m-1}{m-1}=\binom{N-1}{m-1} \quad (5)$$

According to combinatorial relations, we have

$$C_N^m = \frac{N}{m}C_{N-1}^{\text{m-1}} \text{ or } C_{N-1}^{m-1} = \frac{m}{N}C_N^m \quad (6)$$

$$C_N^m - C_{N-1}^m = C_{N-1}^{m-1} (7)$$

Equation (7) presents the recursive relation between the m-dimensional linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N)$ and the $(m-1)$-dimensional linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_{m-1} = N)$. This equation (7) also provides the recursive relation between the height$N$linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N)$ and the height $(N-1)$ linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N-1)$. Equation (7) can be used, based on height N and dimension m, to prove the number of solutions of the linear Diophantine equation by mathematical induction.

Local-Global Principle

The number of integer solutions of equation (1) is denoted as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1} = \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots(N-(m-1))}{N} \quad (8)$$

The product $\frac{1}{m!}N(N-1)(N-2)\cdots(N-(m-1))$ in the formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$, represents the total number of integer solutions of the equation in$m$dimensions.

In the product $\frac{1}{m!}\frac{N(N-1)(N-2)\cdots(N-(m-1))}{N}$, the product $\frac{1}{m!}N(N-1)(N-2)\cdots(N-(m-1))$ is divided by N, therefore for each N $(N = 1,2,\cdots,i,\cdots,N)$, if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ $(1 \le i \le N)$) have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Linear Diophantine equations can provide clear or concise numerical equivalence relations in each local equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = i \ (1 \le i \le N))$.

In linear Diophantine equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$, when $m = 3$, the equation is transformed into $x_1 + x_2 + x_3 = N$.

According to lemma 5, the number of integer solutions to the equation $x_1 + x_2 + x_3 = N$ is equal to the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$.

According to lemma 4, the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$ is equal to the number of integer solutions to the equation $x_1 + x_2 - x_3 = 0$ or $x_1 + x_2 = x_3$.
Now, consider the enumeration method to list the integer solutions of linear Diophantine equations with a countable solution set.

From the above, when $m = 3$, the number of integer solutions to the equation $x_1 + x_2 = x_3$ or $x_1 + x_2 + x_3 = N$ is equal to $C_{N-1}^{3-1} = C_{N-1}^2$.

Table 1 below shows the total number of integer solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \le 10$.

The number of integer solutions to the equation $x_1 + x_2 = x_3$is equal to

$$C_{N-1}^{3-1} = C_{N-1}^2 = \frac{(N-1)(N-2)}{2!} \quad (9)$$

When $N \le 10$, as shown in Table 1.

When $m = 2$, the number of integer solutions to the equation $x_1 + x_2 = N$ is equal to

$$C_N^2 - C_{N-1}^2 = \frac{(N)(N-1)}{2!} - \frac{(N-1)(N-2)}{2!} = C_{N-1}^1 = N-1 \quad (10)$$

is indicated by the diagonal line in Table 1.
Lemma 1 is proved.

Lemma 2 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ be the number of integer solutions of the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

Table 1. Total number of integer solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \le 10$

$x_1 + x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10		
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10			
3	1+3=4	2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=10				
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10					
5	1+5=6	2+5=7	3+5=8	4+5=9	5+5=10						
6	1+6=7	2+6=8	3+6=9	4+6=10							
7	1+7=8	2+7=9	3+7=10								
8	1+8=9	2+8=10									
9	1+9=10										
10											
x_2											

$x_1 = \{1,2,3,\cdots,10\}$, $x_2 = \{1,2,3,\cdots,10\}$ and $x_3 = \{1,2,3,\cdots,10\}$

the approximate number of integer solutions for the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) \sim \frac{1}{(m-1)!}\frac{1}{N}N^m + O(N^{m-2})$$

For any non-negative integer$N$, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Proof

Combinatorial approximation method

According to lemma 1, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$has the solution set

$x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

The number of integer solutions for this equation is $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$.

In the formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$, m is finite and relatively N very small; $N$is very large and infinite.

The formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$ is

$$\frac{N(N-1)\cdots(N-(m-1))}{m!} - \frac{(N-1)(N-2)\cdots(N-m)}{m!} = \frac{(N-1)(N-2)\cdots(N-(m-1))}{(m-1)!} \quad (11)$$

We make the following approximation $N(N-1)\cdots(N-(m-1)) = N^m + O(N^{m-1})$,$(N-1)(N-2)\cdots(N-m) = \langle N-1\rangle^m + O(N^{m-1})$ and $(N-1)(N-2)\cdots(N-(m-1)) = N^{\langle m-1\rangle} + O(N^{m-2})$.

Obtain

$$\frac{N^m}{m!} - \frac{(N-1)^m}{m!} = \frac{N^{(m-1)}}{(m-1)!} + O(N^{m-2}) \quad (12)$$

Local-Global Principle

The approximation number of solutions to equation (1) is denoted by

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \frac{1}{(m-1)!}\frac{N^m}{N} + O(N^{m-2}) \quad (13)$$

The product $\frac{N^m}{m!}$ in the above formula (12) is the total number of solutions for the m-dimensional equation.

In $\frac{N^m}{m!N}$, the product $\frac{N^m}{m!}$ is divided by N, therefore for each N $(N = 1,2,\cdots,i,\cdots,N)$, if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ $(1 \le i \le N)$have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Equation $x_1 + x_2 = x_3$ has the solution set $x_1 = x_2 = x_3 = \{1,2,3,\cdots,N\}$, and the number of solution set is $|x_1| = |x_2| = |x_3| = N$, and the density of the solution set is $d_1 = d_2 = d_3 = 1$.

Table 2 lists the total number of solutions for the 3D equation $x_1 + x_2 = x_3$ satisfies $x_i \le 10$.

When $m = 3$, the number of integer solutions of the equation $x_1 + x_2 = x_3$ or

$x_1 + x_2 + x_3 = N$ is equal to $NS \sim \frac{N^2}{2!}$.

When $m = 2$, the number of integer solutions of the equation $x_1 + x_2 = N$ is equal to

$NS \sim \frac{N^2}{2!} - \frac{(N-1)^2}{2!} \sim \frac{1}{1!}N^1$. shown by the diagonal line in Table 2.

The proof of Lemma 2 is completed.

Lemma 3 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ denote the number of integer solutions of the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

and the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) \sim \frac{1}{(m-1)!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2})$$

For any non-negative integer$N$, the solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$, the solution set can be

$x_i \ne x_j$, and the number of the solution sets is $|x_i| \le N$, and it can be $|x_i| \ne |x_j|$.

Table 2 Total number of solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \le 10$

$x_1 + x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10		
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10	9+2=11		
3	1+3=4	2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=10	8+3=11	9+3=12		
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10	7+4=11	8+4=12	9+4=13		
5	1+5=6	2+5=7	3+5=8	4+5=9	5+5=10	6+5=11	7+5=12	8+5=13	9+5=14		
6	1+6=7	2+6=8	3+6=9	4+6=10	5+6=11	6+6=12	7+6=13	8+6=14	9+6=15		
7	1+7=8	2+7=9	3+7=10	4+7=11	5+7=12	6+7=13	7+7=14	8+7=15	9+7=16		
8	1+8=9	2+8=10	3+8=11	4+8=12	5+8=13	6+8=14	7+8=15	8+8=16	9+8=17		
9	1+9=10	2+9=11	3+9=12	4+9=13	5+9=14	6+9=15	7+9=16	8+9=17	9+9=18		
10											
x_2											

Solution set x_1, $x_2$and x_3 is $x_1 = \{1,2,3,\cdots,10\}$, $x_2 = \{1,2,3,\cdots,10\}$ and $x_3 = \{1,2,3,\cdots,10\}$.

Proof

Combinatorial approximation method

According to lemma 2, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ has the solution set $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$, for the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$, has the following approximate number of integer solutions:

$$NS = \frac{N^m}{m!} - \frac{(N-1)^m}{m!} = \frac{1}{(m-1)!} N^{(m-1)} + O(N^{m-2}) \quad (14)$$

We rewrite the above expression as

$$\frac{N^m}{m!} - \frac{(N-1)^m}{m!} = \frac{1}{(m-1)!}\frac{1}{N} N^m + O(N^{m-2}) \quad (15)$$

The solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$and it can be $x_i \neq x_j$, and the number of the solution set is $|x_i| \leq N$, and it can be $|x_i| \neq |x_j|$, and the density of the solution set is $d_i = \frac{|x_i|}{N}$.

The above expression (15) multiplied by the density $(d_1 \cdot d_2 \cdot \cdots \cdot d_i \cdot \cdots \cdot d_m)$ of the solution sets

$$(d_1 \cdot d_2 \cdot \cdots \cdot d_m)\left\{\frac{N^m}{m!} - \frac{(N-1)^m}{m!}\right\} \sim \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot d_2 \cdot \cdots \cdot d_m)N^m \quad (16)$$

$$\text{Left side} = \frac{(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)}{m!} - \frac{(d_1 \cdot N - d_1)(d_2 \cdot N - d_2)\cdots(d_m \cdot N - d_m)}{m!} \quad (17)$$

Since $0 < d_i \leq 1$, we make the following approximation $(d_1 \cdot N - d_1)(d_2 \cdot N - d_2)\cdots(d_m \cdot N - d_m) \sim (d_1 \cdot N - 1)(d_2 \cdot N - 1)\cdots(d_m \cdot N - 1)$.

The above formula can be approximated as

$$\text{Left side} = \frac{(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)}{m!} - \frac{(d_1 \cdot N - 1)(d_2 \cdot N - 1)\cdots(d_m \cdot N - 1)}{m!}$$

$$\text{Left side} = \frac{|x_1||x_2|\cdots|x_m|}{m!} - \frac{(|x_1|-1)(|x_2|-1)\cdots(|x_m|-1)}{m!}$$

$$\text{Right side} = \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot d_2 \cdot \cdots \cdot d_m)N^m = \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)$$

$$\text{Right side} = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m|$$

Left side = Right side

$$\frac{|x_1||x_2|\cdots|x_m|}{m!} - \frac{(|x_1|-1)(|x_2|-1)\cdots(|x_m|-1)}{m!} = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2}) \quad (18)$$

Consider the non-uniform distribution of the number of integer solutions caused by the combination of density d_i in each m dimension with N change, and then $\frac{1}{(m-1)!}$ is modified to $c(m, d_i)$. $c(m, d_i)$ can be obtained through calculation based on precision; that is

$$N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| = c(m, d_i)\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2}) \quad (19)$$

Local-Global Principle

The approximation number of integer solutions of equation (1) is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) \sim \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2}) \quad (20)$$

The product $\frac{|x_1||x_2|\cdots|x_m|}{m!}$ in the above formula (18) is the total number of integer solutions in dimension m.

In $\frac{|x_1||x_2|\cdots|x_m|}{Nm!}$, the product $\frac{|x_1||x_2|\cdots|x_m|}{m!}$ is divided by N, N ($N = 1,2,\cdots,i,\cdots,N$), if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ $(1 \leq i \leq N)$ have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Example 1. Prime number

Solution set $p = \{2,3, 5,\cdots,p_i,\cdots,N\}$, $p_i \leq N$, the number of prime solution sets is $|p| = \frac{N}{\ln N}$, and the density of the prime solution sets is $d_1 = d_2 = \frac{1}{\ln N}$. The approximate number of integer solutions of the equation $p_1 + p_2 = N$ is $NS \sim c\frac{|p_1|\cdot|p_2|}{1!N} \sim \frac{0.62N}{(\ln N)^2}$ (including duplicate solutions). For details on Goldbach's conjecture and the number of solutions of Goldbach's equation $p_1 + p_2 = N$, please see reference[8].

Table 3. Total number of integer solutions of 2D equation $p_1 + p_2 = N$ where $N \leq 10$

$p_1 + p_2 = N$	2	3	5	7	p_1
2	2+2=4				
3		3+3=6	5+3=8	7+3=10	
5		3+5=8	5+5=10	7+5=12	
7		3+7=10	5+7=12	7+7=14	
p_2					

$x_i = \{2,3, 5,\cdots,p_i,\cdots,N\} \subset \{1,2,3,\cdots,N\}, |x_i| \leq N = 10$

Example 2. Residue class

In the equation $a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_mx_m = N$, the solution set of the residue class is $a_ix_i = \{a_i \cdot 1, a_i \cdot 2, \ldots, a_i \cdot k, \ldots, N\}$.

By substitution $a_ix_i = y_i$,
we can obtain $y_1 + y_2 + \cdots + y_i + \cdots + y_m = N$.

Solution set y_i is $a_ix_i = \{a_i \cdot 1, a_i \cdot 2, \ldots, a_i \cdot k, \ldots, N\}$, therefore, when $a_i \cdot k \leq N$, and the number of the solution set is $|a_ix_i| = \frac{N}{a_i}$.

In the equation $x_1 + 2x_2 = x_3$, the solution sets x_1,$2x_2$ and x_3 are
$x_1 = x_3 = \{1,2,3,\cdots,k,\cdots,N\}$
$2x_2 = \{2,4,\cdots,2k,\cdots,N\}$

The number of the solution set $|x_1| = |x_3| = N, |2x_2| = \frac{N}{2}$.

The number of integer solutions is $NS \sim \frac{|x_1|\cdot|2x_2|\cdot|x_3|}{2!N} = \frac{N \cdot \frac{N}{2} \cdot N}{2N} = \frac{N^2}{4}$.

Table 4. Total number of integer solutions of the equation $x_1 + 2x_2 = x_3$, where $x_{1,\ 3} \le 10, 2x_2 \le 10$

$x_1 + 2x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10	9+2=11		
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10	7+4=11	8+4=12	9+4=13		
6	1+6=7	2+6=8	3+6=9	4+6=10	5+6=11	6+6=12	7+6=13	8+6=14	9+6=15		
8	1+8=9	2+8=10	3+8=11	4+8=12	5+8=13	6+8=14	7+8=15	8+8=16	9+8=17		
10											
$2x_2$											

$x_i \subseteq \{1,2,3,\cdots,N\}$, $x_i \ne x_j, |x_i| \ne |x_j|, |x_i| \le N = 10$

Density Transformation Method

We now discuss the density transformation of linear Diophantine equations whose solution set is an integer subset.

Let a_1,a_2,...,a_m be positive integers to satisfy $gcd(a_1, a_2, \cdots, a_m) = 1$. In addition, let

$NS = N(a_1, a_2, \cdots, a_m; N)$ be the number of integer solutions of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_mx_m = N \quad (21)$$

For non-negative integers x_1,x_2,...,x_m. According to Schur's theorem, the approximate number of integer solutions of equation (21) is

$$NS = N(a_1, a_2, \cdots, a_m; N) = \frac{N^{m-1}}{(m-1)!\cdot a_1\cdot a_2\cdot\cdots\cdot a_m} + O(N^{m-2}) \quad (22)$$

For non-negative integers$N$, provided that N is sufficiently large, especially when there exists an integerN_1such that each $N \ge N_1$ has at least one representation [2].

Equation $$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N \quad (23)$$

Non-negative integers x_1,x_2,...,x_m, and the solution sets are integer subsets.

$x_i = \{(a_i)_1, (a_i)_2, (a_i)_3, \cdots, N\} \subseteq \{1,2,3,\cdots,N\}(1 \le i \le m)$, and can be$x_i \ne x_j$. When $(a_i)_k \le N$, and the density of the solution set is $d_i = \frac{|x_i|}{N}$, and the number of the solution set is $|x_i| = d_i \cdot N \le N$.

Transform $x_i = \frac{1}{d_i} \cdot y_i$, the solution set is $y_i = \{1,2,3,\cdots,N\}(1 \le i \le m)$ is the integer set.

Solution set $\frac{1}{d_i} \cdot y_i = \left\{\frac{1}{d_i} \cdot 1, \frac{1}{d_i} \cdot 2, \dots, \frac{1}{d_i} \cdot k, \dots, N\right\} \subseteq \{1, 2,\ 3, \cdots, k, \cdots, N\}$, and the number of the solution set is$\left|\frac{1}{d_i} \cdot y_i\right| = d_i \cdot N \le N$, the solution set is $\frac{1}{d_i} \cdot y_i$, the density is d_i, and there are

$$\frac{1}{d_1} \cdot y_1 + \frac{1}{d_2} \cdot y_2 + \cdots + \frac{1}{d_i} \cdot y_i + \cdots + \frac{1}{d_m} \cdot y_m = N \quad (24)$$

According to formula (22), the approximate number of solutions of equation (24) is

$$NS = N\left(\frac{1}{d_1}, \frac{1}{d_2}, \cdots, \frac{1}{d_m}; N\right) = \frac{m}{N}\frac{1}{(m)!}(d_1 \cdot N)(d_2 \cdot N) \cdot \dots \cdot (d_m \cdot N) + O(N^{m-2}) \quad (25)$$

where $d_i \cdot N = |x_i|$. The approximate number of solutions of equation (24) is also

$$NS = \frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} + O(N^{m-2}) \quad (26)$$

Lemma 3 is proved.

For further discussion on the cases of unequal density $d_i \cdot N = |x_i|$ and applications, see reference [8-10].

Lemma 4 The number of integer solutions of the equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N$ is equal to the number of integer solutions of the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof

Combinatorial approximation method

In the equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N$, (27)

By substituting $x_1 = y_1, \pm x_i = y_i(2 \le i \le m)$

we can obtain $y_1 + y_2 + \cdots + y_i + \cdots + y_m = N$. (28)

When x_i, $y_i(1 \le i \le m) \subset \{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i \subset \{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N \le -k \le -1$, the number of the solution set is $|x_1| = |y_1|$, $|-x_i| = |x_i| = |y_i|(2 \le i \le m)$.

According to lemma 3, the approximate number of integer solutions of equation (27) can be expressed as

$$NS \sim c\frac{|x_1|\cdot|\pm x_2|\cdots|\pm x_m|}{N\cdot(m-1)!} = c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} + O(N^{m-2}) \quad (29)$$

According to lemma 3, the approximate number of integer solutions of equation (28) can be expressed as

$$NS \sim c\frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!} = c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} + O(N^{m-2}) \quad (30)$$

When x_i, $y_i(1 \le i \le m) = \{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i = \{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N \le -k \le -1$, the number of the solution set is $|x_1| = |y_1| = N$, $|-x_i| = |x_i| = |y_i| = N(2 \le i \le m)$.

According to lemma 2, the approximate number of integer solutions of equation (27) can be expressed as

$$NS \sim c\frac{|x_1|\cdot|\pm x_2|\cdots|\pm x_m|}{N\cdot(m-1)!} = c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} = c\frac{N^{m-1}}{(m-1)!} + O(N^{m-2}) \quad (31)$$

According to lemma 2, the approximate number of integer solutions of equation (28) can be expressed as

$$NS \sim c\frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!} = c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} = c\frac{N^{m-1}}{(m-1)!} + O(N^{m-2}) \quad (32)$$

The number of integer solutions of equation (27) is equal to the number of integer solutions of equation (28).

Proof of Lemma 4 is completed.

Lemma 5 The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof

Combinatorial approximation method

Equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ (33)

becomes $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m + N = N$ (34)

By substitution $x_1 + N = y_1$ and $\pm x_i = y_i (2 \le i \le m)$
we can obtain $y_1 + y_2 + \cdots + y_i + \cdots + y_m = N$. (35)

Since N is a constant; therefore, it is not included the number of the solution set. The number of solution set $|x_1 + N|$ is only regarded as $|x_1 + N| = |x_1| = |y_1|$.

When $x_i,\ y_i (1 \le i \le m) \subset \{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i = \{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N \le -k \le -1$, the number of the solution set is $|-x_i| = |x_i| = |y_i| (2 \le i \le m)$.

According to lemma 3, the approximate number of integer solutions of equation (33) can be expressed as

$$NS \sim c\frac{|x_1+N|\cdot|\pm x_2|\cdots|\pm x_m|}{N\cdot(m-1)!} = c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} + O(N^{m-2}) \quad (36)$$

According to lemma 3, the approximate number of integer solutions of equation (35) can be expressed as

$$NS \sim c\frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!} = c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} + O(N^{m-2}) \quad (37)$$

Since N is a constant; therefore, it is not included the number of the solution set. The number of solution set $|x_1 + N|$ is only regarded as $|x_1 + N| = |x_1| = |y_1| = N$.

When $x_i,\ y_i (1 \le i \le m) = \{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i = \{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N \le -k \le -1$, the number of the solution set is $|-x_i| = |x_i| = |y_i| = N (2 \le i \le m)$.

According to lemma 2, the approximate number of integer solutions of equation (33) can be expressed as

$$NS \sim c\frac{|x_1+N|\cdot|\pm x_2|\cdots|\pm x_m|}{N\cdot(m-1)!} = c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} = c\frac{N^{m-1}}{(m-1)!} + O(N^{m-2}) \quad (38)$$

According to Lemma 2, the approximate number of integer solutions of equation (35) can be expressed as

$$NS \sim c\frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!} = c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} = c\frac{N^{m-1}}{(m-1)!} + O(N^{m-2}) \quad (39)$$

The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof of Lemma 5 is completed.

Enumeration Method of Lemma 4 and Lemma 5

For globe
for $x_1 - x_2 = N$ $(N \ge k \ge 0)$, see total number in Table 5.
for $x_1 + x_2 = N$ $(0 \le k \le N)$, see total number in Table 1.
For local
For $x_1 - x_2 = k$, see diagonal at k in Table 5.
For $x_1 + x_2 = N - k$, see diagonal at $N - k$ in Table 1.

For other applications and validations of the approximate combinatorial method, see reference [8-10].

Proof of Erdos and Turan conjecture on arithmetic progressions

First, we consider the situation of 3-term arithmetic progressions.

Because $A \subset \{1,2,\cdots,N\}$ $(N \ge 2)$ is a set with no non-trivial three-term arithmetic progressions, *i.e.* no solutions to $x + y = 2z$ with $x \ne y$, In the discus on Erdos and Turan conjecture on arithmetic progressions, whether the set $A \subset \{1,2,\cdots,N\}$ $(N \ge 2)$ contains no non-trivial three-term arithmetic progressions is equivalent to whether the corresponding linear Diophantine equation $x + y = 2z$ has no solutions.

Recall Lemma 3. When $m = 3$, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ change into $x_1 + x_2 + x_3 = N$. The approximate number of integer solutions to the equation $x_1 + x_2 + x_3 = N$ is

$$NS \sim \frac{|x_1|\cdot|x_2|\cdot|x_3|}{2!N} + O(N) \quad (40)$$

Table 5. Total number of integer solutions of the 3D equation $x_1 - x_2 = x_3$ where $x_i \le 10$

$x_1 - x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1-1=0	2-1=1	3-1=2	4-1=3	5-1=4	6-1=5	7-1=6	8-1=7	9-1=8		
2		2-2=0	3-2=1	4-2=2	5-2=3	6-2=4	7-2=5	8-2=6	9-2=7		
3			3-3=0	4-3=1	5-3=2	6-3=3	7-3=4	8-3=5	9-3=6		
4				4-4=0	5-4=1	6-4=2	7-4=3	8-4=4	9-4=5		
5					5-5=0	6-5=1	7-5=2	8-5=3	9-5=4		
6						6-6=0	7-6=1	8-6=2	9-6=3		
7							7-7=0	8-7=1	9-7=2		
8								8-8=0	9-8=1		
9									9-9=0		
10											
x_2											

According to Lemma 4, the number of integer solutions to the equation $x_1 + x_2 = x_3$ is equal to the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$.

According to Lemma 5, the number of integer solutions to equation $x_1 + x_2 + x_3 = 0$is equal to the number of integer solutions to the equation $x_1 + x_2 + x_3 = N$, while$N$is a constant of integer.

Transform the equation $x_1 + x_2 = x_3$ to

$$x_1 + x_2 + x_3 = N \tag{41}$$

The approximate number of integer solutions to the equation $x_1 + x_2 + x_3 = N$ or$x_1 + x_2 = x_3$ is

$$NS \sim \frac{|x_1|\cdot|x_2|\cdot|x_3|}{2!N} + O(N) \tag{42}$$

To obtain the approximate number of integer solutions to the equation $x + y = 2z$ with $x \neq y$, we only need the number or density of the solution set x,y and $2z$.

The following is the procedure to solve density.

1. In the different situation (Bloom and Sisask's situation, Behrend's situation, the case of powers of integer), we assume different forms or models of density of the solution sets (integer subsets) or progressions of integer subsets$x$,y and$z$, then to compute the approximate number of integer solutions to the equation$x + y = 2z$ with $x \neq y$.
2. When the equation $x + y = 2z$ with $x \neq y$has no solutions, then the approximate number of integer solutions to the equation $x + y = 2z$ with $x \neq y$ is less than 1.
3. From the condition that the approximate number of integer solutions to the equation $x + y = 2z$ with $x \neq y$ is less than 1, we can get the density of the progressions or integer subsets (the solution sets).

Three-term arithmetic progressions (3-AP)

a) Bloom and Sisask's situation (Similar to case of prime numbers)

Bloom, T. F., and Sisask (2021) improved the upper bound for Roth's theorem on three-term arithmetic progressions in the integers[1].

Theorem (Bloom, T. F., and Sisask) [1]. Let $N \geq 2$ and $A \subset \{1,2,\cdots,N\}$ be a set with no non-trivial three-term arithmetic progressions, *i.e.* no solutions to $x + y = 2z$with $x \neq y$. Then

$$|A| \ll \frac{N}{(log\,N)^{1+c}} \tag{43}$$

where $c > 0$ is an absolute constant.

Let us consider the integer solutions to $x + y = 2z$. Transform it to

$$x_1 + x_2 + 2x_3 = N. \tag{44}$$

Suppose the density $d_{x_1} = d_{x_2} = \frac{1}{(log\,N)^c}$ and $d_{2x_3} = \frac{1}{2(log\,N)^c}$, we have the number of integer solutions of the equation $x_1 + x_2 + 2x_3 = N$

$$NS \sim \frac{1}{2\,!}\frac{\frac{N}{(log\,N)^C}\cdot\frac{N}{(log\,N)^C}\cdot\frac{N}{2(log\,N)^C}}{N} < 1 \tag{45}$$

$$\frac{N^2}{(log\,N)^{3C}} < 1$$

$$2\,log\,N < 3c\,log(log\,N)$$

$$\frac{2}{3}\frac{log\,N}{log(log\,N)} < c \tag{46}$$

When $\frac{2}{3}\frac{log\,N}{log(log\,N)} < c$, then the equation $x_1 + x_2 + 2x_3 = N$ or $x + y = 2z$ has no solutions.

Theorem 2.1 Let $N \geq 2$ and $A \subset \{1,2,\cdots,N\}$ be a set with no non-trivial three-term arithmetic progressions, *i.e.* no solutions to $x + y = 2z$ with $x \neq y$. Then

$$|A| < \frac{N}{(log\,N)^c}$$

where $\frac{2}{3}\frac{log\,N}{log(log\,N)} < c$ is an absolute constant.

When $\frac{2}{3}\frac{log\,N}{log(log\,N)} > c$, $A \subset \{1,2,\cdots,N\}$ includes non-trivial three-term arithmetic progressions. When $c = 1$, the proposition degenerates into the progressions of prime.

b) Behrend's situation

It was shown by Behrend[3] (1946) that for infinitely many values N there are indeed subsets

$A \subseteq [N]$ of density roughly $\delta \approx 2^{-log(N)^{\frac{1}{2}}}$ or $exp\left(-c(log\,N)^{\frac{1}{2}}\right)$ which have no non-trivial 3-term progressions.

Z. Kelley and R. Meka[4,5] (2023) showed that for some constant $\beta > 0$, any subset A of integers$\{1,2,\cdots,N\}$of size at least $2^{-O(log(N)^{\beta})}\cdot N$, $exp\left(-c(log\,N)^{\frac{1}{12}}\right)\cdot N$ or $exp\left(-c(log\,N)^{\frac{1}{11}}\right)\cdot N$ contains a non-trivial three-term arithmetic progression.

Theorem (Z. Kelley and R. Meka) [4,5]. The following holds for some absolute constant exponent $\beta > 0$. Suppose $A \subseteq |N|$has density $\delta = \frac{|A|}{N}$. Then, either Acontains a non-trivial 3-progression, or else $\delta \leq 2^{-O(log(N)^{\beta})}$.

Bloom, T. F., and Sisask (2023) improved Kelley and Meka's conclusion[6,7].

Theorem (Bloom, T. F., and Sisask) [6] If $A \subseteq \{1,2,\cdots,N\}$ has no non-trivial three-term arithmetic progressions then

$$|A| \leq exp\left(-c(log\,N)^{\frac{1}{11}}\right)\cdot N$$

for some constant $c > 0$.

Theorem (Bloom, T. F., and Sisask) [7] If $A \subseteq \{1,2,\cdots,N\}$ has no non-trivial three-term arithmetic progressions then

$$|A| \le exp\left(-c(log\,N)^{\frac{1}{9}}\right)\cdot N$$

for some constant $c > 0$.

Let us consider the integer solutions to $x + y = 2z$. According to Lemma 4 and Lemma 5, transform it to

$$x_1 + x_2 + 2x_3 = N. \quad (47)$$

Suppose the density $d_{x_1} = d_{x_2} = \frac{1}{2^d}$ and $d_{2x_3} = \frac{1}{2\cdot 2^d}$, we have the number of integer solutions of the equation $x_1 + x_2 + 2x_3 = N$

$$NS \sim \frac{1}{2\,!}\frac{\frac{N}{(2)^d}\cdot\frac{N}{(2)^d}\cdot\frac{N}{2\cdot(2)^d}}{N} < 1 \quad (48)$$

$$\frac{N^2}{(2)^{3d}} < 1$$

$$2\,log\,N < 3d\,log\,2$$

$$\frac{log(N)^{\frac{2}{3}}}{log\,2} < d \quad (49)$$

When $\frac{log(N)^{\frac{2}{3}}}{log\,2} < d$, then the equation $x_1 + x_2 + 2x_3 = N$ or $x + y = 2z$ has no solutions.

Suppose the density $d_{x_1} = d_{x_2} = \frac{1}{e^d}$ and $d_{2x_3} = \frac{1}{2\cdot e^d}$, we have the number of integer solutions of the equation $x_1 + x_2 + 2x_3 = N$

$$NS \sim \frac{1}{2\,!}\frac{\frac{N}{(e)^d}\cdot\frac{N}{(e)^d}\cdot\frac{N}{2\cdot(e)^d}}{N} < 1 \quad (50)$$

$$\frac{N^2}{(e)^{3d}} < 1$$

$$log(N)^{\frac{2}{3}} < d. \quad (51)$$

When $log(N)^{\frac{2}{3}} < d$, then the equation $x_1 + x_2 + 2x_3 = N$or $x + y = 2z$has no solutions.

Theorem 2.2 If $A \subseteq \{1,2,\cdots,N\}$has no non-trivial three-term arithmetic progressions, then

$$|A| < exp\left(-c(log\,N)^{\frac{2}{3}}\right)\cdot N$$

for some constant $c > 0$.

When $|A| > exp\left(-c(log\,N)^{\frac{2}{3}}\right)\cdot N$, $A \subset \{1,2,\cdots,N\}$ includes non-trivial three-term arithmetic progressions.

c) The case of powers of integer

Let us consider the solutions to $x + y = 2z$. Transform it to

$$x_1 + x_2 + 2x_3 = N. \quad (52)$$

Suppose the density $d_{x_1} = d_{x_2} = \frac{N^{\frac{1}{n}}}{N} = N^{\frac{1-n}{n}}$ and $d_{2x_3} = \frac{N^{\frac{1-n}{n}}}{2}$, we have the number of integer solutions of the equation $x_1 + x_2 + 2x_3 = N$

$$NS \sim \frac{1}{2\,!}\frac{N^{\frac{1}{n}}\cdot N^{\frac{1}{n}}\cdot\frac{N^{\frac{1}{n}}}{2}}{N} < 1 \quad (53)$$

$$N^{\frac{3-n}{n}} < 1$$

$$3 < n \quad (54)$$

When $3 < n$, then the equation $x_1 + x_2 + 2x_3 = N$ or $x + y = 2z$has no solutions.

Theorem 2.3 Let $N \ge 2$ and $A \subset \{1,2,\cdots,N\}$ be a set with no non-trivial three-term arithmetic progressions, *i.e.*no solutions to $x + y = 2z$with $x \ne y$. Then

$$|A| < N^{\frac{1}{n}}$$

where $n > 3$ is integer.

When$n < 3$,$A \subset \{1,2,\cdots,N\}$ includes non-trivial three-term arithmetic progressions.

Similarly, the conjecture of n-term arithmetic progression can be proven.

n-term arithmetic progressions (n-AP)

The conjecture of n-term arithmetic progressions of the integer. Let $N \ge 2$ and $A \subset \{1,2,\cdots,N\}$ be a set with no non-trivial n-term arithmetic progressions, *i.e.* following equations have no solutions.

$$\left.\begin{array}{c} x_m - x_{m-1} = k_{m-1} \\ x_{m-1} - x_{m-2} = k_{m-2} \\ \cdots \\ x_2 - x_1 = k_1 \end{array}\right\} \quad (55)$$

where k_i $(1 \le i \le m-1)$ are integer constant.

According to Lemma 4, the number of integer solutions of the above equations (55) is equivalent to the number of integer solutions of the following equations.

$$\left.\begin{array}{c} x_m + x_{m-1} = k_{m-1} \\ x_{m-1} + x_{m-2} = k_{m-2} \\ \cdots \\ x_2 + x_1 = k_1 \end{array}\right\} \quad (56)$$

Each equation in the equations (56) is added, and one can obtain

$$x_m + 2x_{m-1} + \cdots + 2x_2 + x_1 = k_{m-1} + k_{m-2} + \cdots + k_1 = k \quad (57)$$

We only need to compute the number of integer solutions of the following equations.

$$x_i + x_{i-1} = k_i \quad (58)$$

$$x_m + 2x_{m-1} + \cdots + 2x_2 + x_1 = k \quad (59)$$

Where k_i,k are integer constant, or

$$x_1 + x_2 = N \tag{60}$$

$$x_m + 2x_{m-1} + \cdots + 2x_2 + x_1 = N \tag{61}$$

where N,m are integer constant.

a) Bloom and Sisask's situation (Similar to case of prime numbers)

Suppose the density $d_{x_1} = d_{x_2} = \frac{1}{(log\,N)^c}$, we have the number of integer solutions of the equation $x_1 + x_2 = N$

$$NS \sim \frac{\frac{N}{(log\,N)^C} \cdot \frac{N}{(log\,N)^C}}{N} < 1 \tag{62}$$

$$\frac{N}{(log\,N)^{2C}} < 1$$

$$log\,N < 2c\,log(log\,N)$$

$$\frac{1}{2}\frac{log\,N}{log(log\,N)} < c \tag{63}$$

When $\frac{1}{2}\frac{log\,N}{log(log\,N)} < c$,then the equation $x_1 + x_2 = N$has no solutions.

Suppose the density $d_{x_1} = d_{x_m} = \frac{1}{(log\,N)^c}$ and $d_{2x_i} = \frac{1}{2(log\,N)^c}$ $(2 \le i \le m-1)$, we have the number of integer solutions of the equation $x_m + 2x_{m-1} + \cdots + 2x_2 + x_1 = N$

$$NS \sim \frac{\frac{N}{(log\,N)^C} \cdot \frac{N^{m-2}}{2^{m-2}(log\,N)^{(m-2)C}} \cdot \frac{N}{(log\,N)^C}}{(m-1)! \cdot N} < 1 \tag{64}$$

$$\frac{1}{2^{m-2}(m-1)!} \cdot \frac{N^{m-1}}{(log\,N)^{mC}} < 1$$

$$\frac{\beta \cdot N^{m-1}}{(log\,N)^{mC}} < 1$$

$$\frac{log\,\beta + (m-1)\,log(N)}{m \cdot log(log\,N)} < c \tag{65}$$

When $\frac{log\,\beta + (m-1)\,log(N)}{m \cdot log(log\,N)} < c$, then the equation $x_m + 2x_{m-1} + \cdots + 2x_2 + x_1 = N$ has no solutions, where $\beta = \frac{1}{2^{m-2}(m-1)!}$.

Theorem 2.4 Let $N \ge 2$ and $A \subset \{1,2,\cdots,N\}$ be a set with no non-trivial n-term arithmetic progressions, then

$$|A| < \frac{N}{(log\,N)^c}$$

where $\frac{log\,\beta + (m-1)\,log(N)}{m \cdot log(log\,N)} < c$ is an absolute constant, where $\beta = \frac{1}{2^{m-2}(m-1)!}$.

When $\frac{log\,\beta + (m-1)\,log(N)}{m \cdot log(log\,N)} > c$, $A \subset \{1,2,\cdots,N\}$ includes non-trivial n-term arithmetic progressions.

When $c = 1$, the proposition degenerates into the n-term progressions of prime.

b) Behrend's situation

Suppose the density $d_{x_1} = d_{x_2} = \frac{1}{2^d}$, we have the number of integer solutions of the equation $x_1 + x_2 = N$

$$NS \sim \frac{\frac{N}{(2)^d} \frac{N}{(2)^d}}{N} < 1 \tag{66}$$

$$\frac{N}{(2)^{2d}} < 1$$

$$log\,N < 2d\,log\,2$$

$$\frac{log(N)^{\frac{1}{2}}}{log\,2} < d \tag{67}$$

When $\frac{log(N)^{\frac{1}{2}}}{log\,2} < d$,then the equation $x_1 + x_2 = N$has no solutions.

Suppose the density $d_{x_1} = d_{x_2} = \frac{1}{e^d}$, we have the number of integer solutions of the equation $x_1 + x_2 = N$

$$NS \sim \frac{\frac{N}{(e)^d} \frac{N}{(e)^d}}{N} < 1 \tag{68}$$

$$\frac{N}{(e)^{2d}} < 1$$

$$log(N)^{\frac{1}{2}} < d \tag{69}$$

When $log(N)^{\frac{1}{2}} < d$, then the equation $x_1 + x_2 = N$ has no solutions.

Suppose the density $d_{x_1} = d_{x_m} = \frac{1}{2^d}$ and $d_{2x_i} = \frac{1}{2 \cdot 2^d}$ $(2 \le i \le m-1)$, we have the number of integer solutions of the equation $x_m + 2x_{m-1} + \cdots + 2x_2 + x_1 = N$

$$NS \sim \frac{1}{(m-1)\ !} \frac{\frac{N}{(2)^d} \cdot \frac{N^{m-2}}{2^{m-2} \cdot (2)^{(m-2)d}} \cdot \frac{N}{(2)^d}}{N} < 1 \tag{70}$$

$$\frac{1}{2^{m-2}(m-1)\ !} \frac{N^{m-1}}{\cdot (2)^{md}} < 1$$

$$\beta \cdot N^{m-1} < (2)^{md}$$

$$log\,\beta + (m-1) \cdot log\,N < md\,log(2)$$

$$\frac{log\,\beta + (m-1) \cdot log\,N}{m\,log(2)} < d \tag{71}$$

When $\frac{log\,\beta + (m-1) \cdot log\,N}{m\,log(2)} < d$, then the equation $x_m + 2x_{m-1} + \cdots + 2x_2 + x_1 = N$has no solutions.

Suppose the density $d_{x_1} = d_{x_m} = \frac{1}{e^d}$ and $d_{2x_i} = \frac{1}{2 \cdot e^d}$ $(2 \le i \le m-1)$, we have the number of integer solutions of the equation $x_m + 2x_{m-1} + \cdots + 2x_2 + x_1 = N$

$$NS \sim \frac{1}{(m-1)\ !} \frac{\frac{N}{(e)^d} \cdot \frac{N^{m-2}}{2^{m-2} \cdot (e)^{(m-2)d}} \cdot \frac{N}{(e)^d}}{N} < 1 \tag{72}$$

$$\frac{1}{2^{m-2}(m-1)\,!}\frac{N^{m-1}}{\cdot(e)^{md}}<1$$

$$\beta\cdot N^{m-1}<(e)^{md}$$
$$\log\beta+(m-1)\cdot\log N<md\log(e)$$

$$\frac{\log\beta+(m-1)\cdot\log N}{m}<d \tag{73}$$

When $\frac{\log\beta+(m-1)\cdot\log N}{m}<d$, then the equation $x_m+2x_{m-1}+\cdots+2x_2+x_1=N$ has no solutions.

Theorem 2.5 If $A\subseteq\{1,2,\cdots,N\}$ contains no non-trivial n-term arithmetic progressions, then

$$|A|<exp\left(-c(\log N)^{\frac{(m-1)}{m}}\right)\cdot N$$

for some constant $c>0$.

When $|A|>exp\left(-c(\log N)^{\frac{(m-1)}{m}}\right)\cdot N,\quad A\subset\{1,2,\cdots,N\}$ includes non-trivial n-term arithmetic progressions.

c) The case of powers of integer

Let us consider the solutions to $x_1+x_2=N$.

Suppose the density $d_{x_1}=d_{x_2}=\frac{N^{\frac{1}{n}}}{N}=N^{\frac{1-n}{n}}$, we have the number of integer solutions of the equation $x_1+x_2=N$

$$NS\sim\frac{N^{\frac{1}{n}}\cdot N^{\frac{1}{n}}}{N}<1 \tag{74}$$

$$N^{\frac{2-n}{n}}<1$$
$$n>2$$

When $n>2$, then the equation $x_1+x_2=N$ has no solutions.

Suppose the density $d_{x_1}=d_{x_m}=\frac{N^{\frac{1}{n}}}{N}=N^{\frac{1-n}{n}}$ and $d_{2x_i}=\frac{N^{\frac{1-n}{n}}}{2}(2\le i\le m-1)$, we have the number of integer solutions of the equation $x_m+2x_{m-1}+\cdots+2x_2+x_1=N$

$$NS\sim\frac{1}{(m-1)\,!}\frac{N^{\frac{1}{n}}\cdot N^{\frac{1}{n}}\cdot(\frac{N^{\frac{1}{n}}}{2})^{m-2}}{N}<1 \tag{75}$$

$$\frac{1}{2^{m-2}(m-1)\,!}N^{\frac{m-n}{n}}<1$$

$$N^{\frac{m-n}{n}}<1$$

$$m<n \tag{76}$$

When $m<n$, then the equation $x_m+2x_{m-1}+\cdots+2x_2+x_1=N$ has no solutions.

Theorem 2.6 Let $N\ge 2$ and $A\subset\{1,2,\cdots,N\}$ be a set with no non-trivial n-term arithmetic progressions, then

$$|A|<N^{\frac{1}{n}}$$

where $m<n$ is integer.

When $m>n$, $A\subset\{1,2,\cdots,N\}$ includes non-trivial n-term arithmetic progressions.

32-term arithmetic progressions (2-AP)

2-term arithmetic progressions for the integer. Let $N\ge 2$ and $A\subset\{1,2,\cdots,N\}$ be a set with no non-trivial 2-term arithmetic progressions, *i.e.* the equation $x_1\pm x_2=N$ has no solutions.

We only need to compute the number of integer solutions of the following equations,

$$x_1\pm x_2=N \tag{77}$$

where N is integer constant.

a) Bloom and Sisask's situation (Similar to case of prime numbers)

Suppose the density $d_{x_1}=d_{x_2}=\frac{1}{(\log N)^c}$, we have the number of integer solutions of the equation $x_1\pm x_2=N$

$$NS\sim\frac{\frac{N}{(\log N)^C}\frac{N}{(\log N)^C}}{N}<1 \tag{78}$$

$$\frac{N}{(\log N)^{2C}}<1$$
$$\log N<2c\log(\log N)$$
$$\frac{1}{2}\frac{\log N}{\log(\log N)}<c \tag{79}$$

When $\frac{1}{2}\frac{\log N}{\log(\log N)}<c$, then the equation $x_1\pm x_2=N$ has no solutions.

Theorem 2.7 Let $N\ge 2$ and $A\subset\{1,2,\cdots,N\}$ be a set with no non-trivial 2-term arithmetic progressions, then

$$|A|<\frac{N}{(\log N)^c}$$

where $\frac{1}{2}\frac{\log N}{\log(\log N)}<c$ is an absolute constant.

When $\frac{1}{2}\frac{\log N}{\log(\log N)}>c$, $A\subset\{1,2,\cdots,N\}$ includes non-trivial 2-term arithmetic progressions.

When $c=1$, the proposition degenerates into the 2-term progressions of prime, or strong Goldbach's conjecture and the conjecture of twin prime numbers.

b) Behrend's situation

Suppose the density $d_{x_1}=d_{x_2}=\frac{1}{2^d}$, we have the number of integer solutions of the equation $x_1\pm x_2=N$

$$NS\sim\frac{\frac{N}{(2)^d}\frac{N}{(2)^d}}{N}<1 \tag{80}$$

$$\frac{N}{(2)^{2d}}<1$$
$$\log N<2d\log 2$$

$$\frac{\log(N)^{\frac{1}{2}}}{\log 2} < d \quad (81)$$

When $\frac{\log(N)^{\frac{1}{2}}}{\log 2} < d$, then the equation $x_1 \pm x_2 = N$has no solutions.

Suppose the density $d_{x_1} = d_{x_2} = \frac{1}{e^d}$, we have the number of integer solutions of the equation $x_1 \pm x_2 = N$

$$NS \sim \frac{\frac{N}{(e)^d}\frac{N}{(e)^d}}{N} < 1 \quad (82)$$

$$\frac{N}{(e)^{2d}} < 1$$

$$\log(N)^{\frac{1}{2}} < d \quad (83)$$

When $\log(N)^{\frac{1}{2}} < d$,then the equation $x_1 \pm x_2 = N$has no solutions.

Theorem 2.8 If $A \subseteq \{1,2,\cdots,N\}$ contains no non-trivial 2-term arithmetic progressions, then

$$|A| < exp\left(-c(\log N)^{\frac{1}{2}}\right) \cdot N$$

for some constant $c > 0$.

When$|A| > exp\left(-c(\log N)^{\frac{1}{2}}\right) \cdot N, A \subset \{1,2,\cdots,N\}$ includes non-trivial 2-term arithmetic progressions.

c) The case of powers of integer

Let us consider the solutions to $x_1 \pm x_2 = N$.

Suppose the density $d_{x_1} = d_{x_2} = \frac{N^{\frac{1}{n}}}{N} = N^{\frac{1-n}{n}}$,we have the number of integer solutions of the equation $x_1 \pm x_2 = N$

$$NS \sim \frac{N^{\frac{1}{n}} \cdot N^{\frac{1}{n}}}{N} < 1 \quad (84)$$

$$N^{\frac{2-n}{n}} < 1$$

$$n > 2$$

When $n > 2$, then the equation $x_1 \pm x_2 = N$ has no solutions.

Theorem 2.9 Let $N \geq 2$ and $A \subset \{1,2,\cdots,N\}$ be a set with no non-trivial 2-term arithmetic progressions, then

$$|A| < N^{\frac{1}{n}}$$

Where $n > 2$ is integer.

When $n < 2$, $A \subset \{1,2,\cdots,N\}$ includes non-trivial 2-term arithmetic progressions.

Nevanlinna theory deals with the growth of zeros and poles (roots or solution sets) of meromorphic functions. According to Vojta's conjecture, there is a corresponding (similar) relationship between Nevanlinna theory in function fields and the abc conjecture in number fields (including integers). Since the general Fermat conjecture can be derived from the abc conjecture and concerns the number of zeros (roots or solutions) of equations, Nevanlinna theory, the abc conjecture, Vojta's conjecture, and the general Fermat conjecture all of those probloms is the same class of problems related to the number of zeros (roots or solutions) of equations.

This article presents an approximate combinatorial method for estimating the number of integer solutions to general Diophantine equations with countable solution sets. The applicable countable solution sets include integers, rational numbers, prime numbers, algebraic numbers, polynomials, algebraic number fields, and function fields. Some problems can also be transformed into counting number of solutions of equations. For applications of this method in other areas, such as the essence and number of solutions of the abc conjecture, the number of solutions to the general Fermat conjecture, and so forth, please refer to the reference [8].

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