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Research Article

A CONCISE PROOF OF GOLDBACH CONJECTURE AND HARDY-LITTLEWOOD PRIME SUM CONJECTURE

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Abstract

A concise proof of the Goldbach Conjecture and the Hardy-Littlewood Prime Sum Conjecture are given in this paper. The core idea of this paper is to transform the problem of representing an integer as a sum of several primes into counting the number of integer solutions of the corresponding Diophantine equation. Firstly, the combinatorial approximation method to compute the approximate number of integer solutions to linear Diophantine equation with integer subsets (prime number) as solution set is presented. Secondly, the approximate number of integer solutions of Chen Jingrun's "1+2" theorem and Vinogradov's theorem of sum of three primes are computed to validate the combinatorial approximation method. Finally, the approximate number of integer solutions of the Goldbach (equation) Conjecture and the Hardy-Littlewood Prime Sum (equation) Conjecture are computed and the results we obtained are consistent with the conjecture, thus proof of the Goldbach Conjecture and the Hardy-Littlewood Prime Sum Conjecture are provided.

Keywords: Goldbach Conjecture, Hardy-Littlewood Prime Sum Conjecture, Combinatorial approximation method, Linear Diophantine equation, Number of solutions, Set of solutions, Subset of integer, Prime number.

INTRODUCTION

Introduction and Results

The strong Goldbach conjecture asserts that every even integer $N > 2$ can be represented as the sum of two prime numbers $N = p_1 + p_2$. Proposed in 1742, this conjecture remains a core open problem in number theory. In 1966, Chen Jingrun, using the sieve method, proved that sufficiently large even integers can be represented as the sum of a prime and the product of at most two primes $N = p_1 + p_2 \cdot p_3$, where p_1,p_2,p_3 are primes. The asymptotic formula for the number of such representations is: for sufficiently large even $N > 2$,

$$r_{1+2}(N) \sim 0.67 \cdot c \cdot \mathfrak{S}(N) \frac{N}{(logN)^2} \tag{1}$$

where $\mathfrak{S}(N) = \prod_{p>2}(1 - \frac{1}{(p-1)^2}) \prod_{p>2,p|N} \frac{p-1}{p-2}$

This result remains the best progress on the strong Goldbach conjecture to date[1, 2].

The weak Goldbach conjecture states that every odd integer $N > 5$ can be represented as the sum of three prime numbers $N = p_1 + p_2 + p_3$. In 1937, I. M. Vinogradov, using the circle method, proved thatsufficiently large odd integers can be represented as the sum of three primes, and gave the asymptotic formula for the number of representations: for sufficiently large odd integer, the number of representations is[3, 4]

$$\mathrm{r}_3(\mathrm{N}) \sim \mathfrak{S}(\mathrm{N}) \frac{\mathrm{N}^2}{2(\ln \mathrm{N})^3} \tag{2}$$

where$\mathfrak{S}(N) = \prod_{p>2,p|N}(1 - \frac{1}{(p-1)^2}) \prod_{p>2} \left(1 + \frac{1}{(p-1)^3}\right)$

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In 2013, H. Helfgott improved Vinogradov's method and combining largescale computer verification, proved that all odd integers greater than 5 satisfy the weak Goldbach conjecture[5].

As a systematic generalization of the Goldbach conjecture, in 1923, G. H. Hardy and J. E. Littlewood [6,7] proposed the prime sum conjecture: for a given integer $k \geq 1$, when$N$is sufficiently large, N has the same parity as$k$,Ncan be represented as the sum of k prime numbers, and gave the asymptotic formula for the number of representations $N = p_1 + p_2 + \cdots + p_k$:

$$r_k(N) \sim c\mathfrak{S}(N) \frac{N^{k-1}}{(k-1)!(ln\, N)^k} \tag{3}$$

where $\mathfrak{S}(N) = \prod_{p>2,p|N}(1 - \frac{1}{(p-1)^{k-1}}) \prod_{p>2} \left(1 + \frac{(-1)^{k-1}}{(p-1)^k}\right)$

In terms of existence proof, in 1930, L. G. Schnirelmannused elementary density theory to prove that there exists a fixed integer$k$ such that every sufficiently large integer can be represented as the sum of$k$ prime numbers[8].Although no asymptotic formula was given, this was the first finite answer to the existence of prime sum problems. The Hardy-Littlewood conjecture and Schnirelmann's work together promoted the development of prime sum problems. When$k = 2$, the conjecture is the strong Goldbach conjecture; when$k = 3$, it corresponds to the asymptotic part of the weak Goldbach conjecture, providing a unified theoretical framework for prime sum problems.

When studying the Goldbach conjecture with Schnirelmann's elementary density theory, one can see that the method suffers from insufficient precision due to inherent constraints in the definition of density. The density-based argument only proves the existence of a finite constant$k$ such that any integer can be represented as a sum of at most k primes. It cannot reduce$k$ to 3 or 2, so it is incapable of proving both the weak and strong

Goldbach conjectures. To address this issue, this paper constructs a combinatorialapproximation approach, converting qualitative estimation at the density level into a counting problem of solution numbers, and derives the asymptotic formula by evaluating the number of solutions to the linear Diophantine equation for sums of$k$ primes.The core idea of this paper is to transform the problem to obtain the asymptotic formula of representing an integer as a sum of several primes into counting the number of integer solutions of the corresponding Diophantine equation containing prime number. To obtain integer solutions to general Diophantine equations is very difficult, whereas determining the number of integer solutions to such equations is comparatively much easier. Combinatorial methods, combinatorial approximation methods, enumeration methods, and the density transformation method can be employed to determine the number of integer solutions to the Diophantine equations with countable solution sets. By using the combinatorial approximation method, the following results can be established unconditionally.

Theorem Equation $p_1 + p_2 + \cdots + p_k = N$, where N, k is integer, $k \geq 1, N \geq N_1 N_1$is suitably large integer,N has the same parity as k,p_i $(1 \leq i \leq k)$ are prime numbers. The number of integer solutions of the equation $p_1 + p_2 + \cdots + p_k = N$ is

$$\frac{1}{(\mathrm{k}-1)\,!}\frac{\mathrm{N}^{\mathrm{k}-1}}{(\ln \mathrm{N})^{\mathrm{k}}} + O(N^{k-2}) = c\frac{N^{k-1}}{(ln\, N)^{k}} + O(N^{k-2}) \tag{4}$$

The number of integer solutions of the equation $p_1 + p_2 + \cdots + p_k = N$ isequivalent to the asymptotic formula for the number of representation $\mathrm{s} = p_1 + p_2 + \cdots + p_k$. When $k = 2$, it corresponds to the strong Goldbach conjecture.

For determining the number of integer solutions of the Diophantine equation, first, we introduce the combinatorial method, approximate combinatorial method, and enumeration method to compute the number of integer solutions of linear Diophantine equations whose solution sets are integer set (integer sequence) and is countable.

Second, for cases where the solution sets and the number of solution sets of the variables in linear Diophantine equations are either the same or different, we propose the approximate combinatorial method, enumeration method, and density transformation method to compute an approximate number of integer solutions of the linear Diophantine equations whose solution sets are integer subsets or sequences of integer subsets.

METHOD AND PROOF OF LEMMA

Lemma

Why study linear Diophantine equations? More specifically, why analyze the number of integer solutions of linear Diophantine equations whose solution sets consist of sequences of integer subsets?

1. The number of integer solutions to linear Diophantine equations can be determined by combinatorial methods.
2. To find integer solutions to nonlinear or higher-order Diophantine equations is very challenging, whereas counting the number of integer solutions for linear Diophantine equations is comparatively easier.
3. The problem of counting the number of integer solutions to nonlinear or higher-order Diophantine equations can be transformed into the problem of counting the number of integer solutions to linear Diophantine equations whose solution sets are integer subsets.
4. The enumeration method allows for straightforward listing of all integer solutions of linear Diophantine equations when the solution set is an integer sequence, or all of integer solutions are expressed as numerical equations or equalities.
5. Since all integer solutions of linear Diophantine equations with solution sets formed by sequences of integer subsets belong to the integer solutions of linear Diophantine equations with integer sequences as solution sets, the enumeration method likewise facilitates the straightforward listing of all such solutions as numerical equations or equalities.
6. By utilizing the enumeration method, it is straightforward to list all local integer solutions, or all of equations or equalities of local integer solutions of linear Diophantine equations whose solution sets are sequences of integer subsets. Specifically, linear Diophantine equations can explicitly or concisely be represented by the numerical equivalence relations within each local equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = i \ (1 \leq i \leq N))$.

This section mainly discusses the number of integer solutions to linear Diophantine equations whose countable solution sets are given by integer sets or integer subset (sequences). The definitions and notations used herein are consistent with those in reference [9].

Lemma 1 presents the number of integer solutions of linear Diophantine equations with integer set sequence as the solution set. Lemma 2 provides the approximate number of integer solutions of linear Diophantine equations with integer set sequence as the solution set. Lemma 3 provides the approximate number of integer solutions of linear Diophantine equations when the solution sets and the number of the solution sets of variables are either the same or different. Lemma 4 and Lemma 5 present the properties of general linear Diophantine equations when transformed into standard linear Diophantine equations.

Lemma 1 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

The number of integer solutions is well known as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1}$$
$$= \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots\big(N-(m-1)\big)}{N}$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Lemma 2 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$

then the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \frac{1}{(m-1)!}\frac{1}{N}N^m + O(N^{m-2})$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Lemma 3 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ be the number of integer solutionsof the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

then the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) \sim \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2})$$

For any non-negative integer N, the solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$, the solution set can be $x_i \neq x_j$, and the number of the solution set is $|x_i| \leq N$, it can be$|x_i| \neq |x_j|$.

Lemma 4 The number of integer solutions of equation$x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Lemma 5 The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof of Lemma

Lemma 1 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

The number of integer solutions is well known as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1}$$
$$= \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots\big(N-(m-1)\big)}{N}$$

For any non-negative integer$N$, the solution set $x_i = \{1,2,3,\cdots,N\}$, the number of the solution set $|x_i| = N$.

Proof

We consider the number of integer solutions of the linear Diophantine equation where the solution set is an integer set or a sequence of integer set, and a countable solution set. The combinatorial method and the enumeration method are applied.

Combinatorial Method

Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N \tag{5}$$

It is well known that the number of integer solutions is[10]

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \binom{N+m-1}{m-1} = C_{N+m-1}^{m-1} \tag{6}$$

For any non-negative integer N, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Substitute $x_i - 1 = y_i$into equation (5) to obtain

$$y_1 + y_2 + \cdots + y_i + \cdots + y_m = N - m \tag{7}$$

The number of integer solutions of equation (7) is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1} \tag{8}$$

The number of integer solutions of equation (7) is equal to the number of integer solutions of equation (5)

C_{m+N-1}^{m-1}=C_{N-1}^{m-1} or $\binom{N+m-1}{m-1}$=$\binom{N-1}{m-1}$(9)

According to combinatorial relations, we have

$$C_N^m = \frac{N}{m}C_{N-1}^{\text{m-1}} \text{ or } C_{N-1}^{m-1} = \frac{m}{N}C_N^m \tag{10}$$

$C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$(11)

Equation (11) presents there cursive relation between the m-dimensional linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N)$ and the $(m-1)$-dimensional linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_{m-1} = N)$. This equation (11) also provides the recursive relation between the height N linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N)$ and the height $(N-1)$ linear Diophantine equation $(x_1 + x_2 + \cdots + x_i + \cdots + x_m = N-1)$. Equation (11) can be used, based on height N and dimension$m$, to prove the number of integer solutions of the linear Diophantine equation by mathematical induction.

Local-Global Principle

The number of integer solutions of equation (5) is denoted as

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \binom{N-1}{m-1} = C_{N-1}^{m-1} = \frac{1}{(m-1)!}\frac{N(N-1)(N-2)\cdots\big(N-(m-1)\big)}{N} \tag{12}$$

The product $\frac{1}{m!}N(N-1)(N-2)\cdots\big(N-(m-1)\big)$ in the formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$ represents the total number of integer solutions of the equation in m dimensions.

In the produc t$\frac{1}{m!}\frac{N(N-1)(N-2)\cdots(N-(m-1))}{N}$, the product $\frac{1}{m!}N(N-1)(N-2)\cdots(N-(m-1))$ is divided by N, therefore for each N ($N = 1,2,\cdots,i,\cdots,N$), if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ ($1 \le i \le N$)) have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Linear Diophantine equations can provide clear or concise numerical equivalence relations in each local equation ($x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ ($1 \le i \le N$)).

In linear Diophantine equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$, when $m = 3$, the equation is transformed into $x_1 + x_2 + x_3 = N$.

According to Lemma 5, the number of integer solutions to the equation $x_1 + x_2 + x_3 = N$ is equal to the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$.

According to Lemma 4, the number of integer solutions to the equation $x_1 + x_2 + x_3 = 0$ is equal tothe number of integer solutions to the equation $x_1 + x_2 - x_3 = 0$ or $x_1 + x_2 = x_3$.

Now, consider the enumeration method to list the integer solutions of linear Diophantine equations with a countable solution set.

From the above, when $m = 3$, the number of integer solutions to the equation $x_1 + x_2 = x_3$or $x_1 + x_2 + x_3 = N$ is equal to $C_{N-1}^{3-1} = C_{N-1}^2$.

Table 1 below shows the total number of integer solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \le 10$.

Lemma 2 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right)$ be the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

the approximate number of integer solutions for the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m}; N\right) = \frac{1}{(m-1)!}\frac{1}{N}N^m + O(N^{m-2})$$

For any non-negative integer$N$, the solution set is $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

Proof

Combinatorial approximation method

According to Lemma 1, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ has the solution set $x_i = \{1,2,3,\cdots,N\}$, and the number of the solution set is $|x_i| = N$.

The number of integer solutions for this equation is $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$.

In the formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$, m is finite and relatively$N$very small; $N$is very large and infinite.

The formula $C_N^m - C_{N-1}^m = C_{N-1}^{m-1}$ is

$$\frac{N(N-1)\cdots(N-(m-1))}{m!} - \frac{(N-1)(N-2)\cdots(N-m)}{m!} = \frac{(N-1)(N-2)\cdots(N-(m-1))}{(m-1)!} \tag{15}$$

Table 1. Total number of integer solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \le 10$

$x_1 + x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10		
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10			
3	1+3=4	2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=10				
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10					
5	1+5=6	2+5=7	3+5=8	4+5=9	5+5=10						
6	1+6=7	2+6=8	3+6=9	4+6=10							
7	1+7=8	2+7=9	3+7=10								
8	1+8=9	2+8=10									
9	1+9=10										
10											
x_2											

$x_1 = \{1,2,3,\cdots,10\}$,$x_2 = \{1,2,3,\cdots,10\}$ and $x_3 = \{1,2,3,\cdots,10\}$

The number of integer solutions to the equation $x_1 + x_2 = x_3$ is equal to

$$C_{N-1}^{3-1} = C_{N-1}^2 = \frac{(N-1)(N-2)}{2!} \tag{13}$$

When $N \le 10$, as shown in Table 1.

When $m = 2$, the number of integer solutions to the equation $x_1 + x_2 = N$ is equal to

$$C_N^2 - C_{N-1}^2 = \frac{(N)(N-1)}{2!} - \frac{(N-1)(N-2)}{2!} = C_{N-1}^1 = N - 1 \tag{14}$$

is indicated by the diagonal in Table 1.
Lemma 1 is proved.

We make the following approximation $N(N-1)\cdots(N-(m-1)) = N^m + O(N^{m-1})$,$(N-1)(N-2)\cdots(N-m) = \langle N-1\rangle^m + O(N^{m-1})$ and $(N-1)(N-2)\cdots(N-(m-1)) = N^{\langle m-1\rangle} + O(N^{m-2})$.

Obtain

$$\frac{N^m}{m!} - \frac{(N-1)^m}{m!} = \frac{N^{(m-1)}}{(m-1)!} + O(N^{m-2}) \tag{16}$$

Local-Global Principle

The approximation number of solutions to equation (5) is denoted by

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \frac{1}{(m-1)!}\frac{N^m}{N} + O(N^{m-2}) \qquad (17)$$

The product $\frac{N^m}{m!}$ in the above formula (16) is the total number of solutions for the m-dimensional equation.

In $\frac{N^m}{m!N}$, the product $\frac{N^m}{m!}$ is divided by N, therefore for each N $(N = 1,2,\cdots,i,\cdots,N)$, if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i$ $(1 \le i \le N)$ have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Equation $x_1 + x_2 = x_3$ has the solution set $x_1 = x_2 = x_3 = \{1,2,3,\cdots,N\}$, and the number of solution set is $|x_1| = |x_2| = |x_3| = N$, and the density of the solution set is $d_1 = d_2 = d_3 = 1$.

Table 2 lists the total number of solutions for the 3D equation $x_1 + x_2 = x_3$ satisfies $x_i \le 10$.

Table 2. Total number of solutions of the 3D equation $x_1 + x_2 = x_3$ where $x_i \le 10$

$x_1 + x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1+1=2	2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=10		
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10	9+2=11		
3	1+3=4	2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=10	8+3=11	9+3=12		
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10	7+4=11	8+4=12	9+4=13		
5	1+5=6	2+5=7	3+5=8	4+5=9	5+5=10	6+5=11	7+5=12	8+5=13	9+5=14		
6	1+6=7	2+6=8	3+6=9	4+6=10	5+6=11	6+6=12	7+6=13	8+6=14	9+6=15		
7	1+7=8	2+7=9	3+7=10	4+7=11	5+7=12	6+7=13	7+7=14	8+7=15	9+7=16		
8	1+8=9	2+8=10	3+8=11	4+8=12	5+8=13	6+8=14	7+8=15	8+8=16	9+8=17		
9	1+9=10	2+9=11	3+9=12	4+9=13	5+9=14	6+9=15	7+9=16	8+9=17	9+9=18		
10											
x_2											

Solution set $x_1 = \{1,2,3,\cdots,10\}$, $x_2 = \{1,2,3,\cdots,10\}$ and $x_3 = \{1,2,3,\cdots,10\}$

When $m = 3$, the number of integer solutions of the equation $x_1 + x_2 = x_3$ or $x_1 + x_2 + x_3 = N$ is equal to $NS \sim \frac{N^2}{2!}$.

When $m = 2$, the number of integer solutions of the equation $x_1 + x_2 = N$ is equal to $NS \sim \frac{N^2}{2!} - \frac{(N-1)^2}{2!} \sim \frac{1}{1!}N^1$. shown by the diagonal line in Table 2.

The proof of Lemma 2 is completed.

Lemma 3 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ denote the number of integer solutions of the standard linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

and the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2})$$

For any non-negative integer N, the solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$, the solution set can be $x_i \ne x_j$, and the number of the solution sets is $|x_i| \le N$, and it can be $|x_i| \ne |x_j|$.

Proof

Combinatorial approximation method

According to lemma 2, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ has the solution set $x_i = \{1,2,3,\cdots,N\}$ and the number of the solution set is $|x_i| = N$. For the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$, one has the following approximate number of integer solutions:

$$NS \sim \frac{N^m}{m!} - \frac{(N-1)^m}{m!} = \frac{1}{(m-1)!}N^{(m-1)} + O(N^{m-2}) \qquad (18)$$

We rewrite the above expression as

$$\frac{N^m}{m!} - \frac{(N-1)^m}{m!} = \frac{1}{(m-1)!}\frac{1}{N}N^m + O(N^{m-2}) \qquad (19)$$

The solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$ and it can be $x_i \ne x_j$, and the number of the solution set is $|x_i| \le N$, and it can be $|x_i| \ne |x_j|$, and the density of the solution set is $d_i = \frac{|x_i|}{N}$.

The above expression (19) multiplied by the density $(d_1 \cdot d_2 \cdot \cdots \cdot d_i \cdot \cdots \cdot d_m)$ of the solution sets

$$(d_1 \cdot d_2 \cdot \cdots \cdot d_m)\left\{\frac{N^m}{m!} - \frac{(N-1)^m}{m!}\right\} \sim \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot d_2 \cdot \cdots \cdot d_m)N^m \qquad (20)$$

$$\text{Left side} = \frac{(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)}{m!} - \frac{(d_1 \cdot N - d_1)(d_2 \cdot N - d_2)\cdots(d_m \cdot N - d_m)}{m!} \qquad (21)$$

Since $0 < d_i \le 1$, we make the following approximation $(d_1 \cdot N - d_1)(d_2 \cdot N - d_2)\cdots(d_m \cdot N - d_m) \sim (d_1 \cdot N - 1)(d_2 \cdot N - 1)\cdots(d_m \cdot N - 1)$.

The above formula can be approximated as

$$\text{Left side} = \frac{(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)}{m!} - \frac{(d_1 \cdot N - 1)(d_2 \cdot N - 1)\cdots(d_m \cdot N - 1)}{m!}$$

$$\text{Left side} = \frac{|x_1||x_2|\cdots|x_m|}{m!} - \frac{(|x_1|-1)(|x_2|-1)\cdots(|x_m|-1)}{m!}$$

$$\text{Right side} = \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot d_2 \cdot \cdots \cdot d_m)N^m = \frac{1}{(m-1)!}\frac{1}{N}(d_1 \cdot N)(d_2 \cdot N)\cdots(d_m \cdot N)$$

$$\text{Right side} = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m|$$

Left side = Right side

$\frac{|x_1||x_2|\cdots|x_m|}{m!} - \frac{(|x_1|-1)(|x_2|-1)\cdots(|x_m|-1)}{m!} = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2})$ (22)

Consider the non-uniform distribution of the number of integer solutions caused by the combination of density d_i in each $m$dimension with N change, and then $\frac{1}{(m-1)!}$ is modified to$c(m,d_i)$. $c(m,d_i)$ can be obtained through calculation based on precision; that is

$$N\left(\underbrace{1,1,\cdots,1}_{m};N\right) \sim \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| = c(m,d_i)\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2}) \quad (23)$$

Local-Global Principle

The approximate number of integer solutions of equation (5) is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2}) \quad (24)$$

The product $\frac{|x_1||x_2|\cdots|x_m|}{m!}$ in the above formula (22) isthe total number of integer solutions in dimension$m$.

In $\frac{|x_1||x_2|\cdots|x_m|}{Nm!}$, the product $\frac{|x_1||x_2|\cdots|x_m|}{m!}$ is divided by N,N ($N = 1,2,\cdots,i,\cdots,N$), if the local equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = i\ (1 \le i \le N)$have or not solutions, then the global equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ also has or not solutions, and vice versa.

Enumeration Method

Example 1. Prime number

Solution set $p = \{2, 3, 5, \cdots, p_i, \cdots, N\}$, $p_i \le N$, the number of prime solution sets is $|p| = \frac{N}{\ln N}$, and the density of the prime solution sets is $d_1 = d_2 = \frac{1}{\ln N}$. The approximate number of integer solutions of the equation $p_1 + p_2 = N$ is $NS \sim c\frac{|p_1|\cdot|p_2|}{1!N} \sim \frac{0.62N}{(\ln N)^2}$ (including duplicate solutions). For original and systematic ideason the number of solutions of the equation with countable solution set and the number of solutions of Goldbach's equation $p_1 + p_2 = N$, please see reference [9].

Example 2. Residue class

In the equation $a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_mx_m = N$, the solution set of the residue class is $a_ix_i = \{a_i \cdot 1, a_i \cdot 2, \ldots, a_i \cdot k, \ldots, N\}$.

Table 3. Total number of integer solutions of 2D equation $p_1 + p_2 = N$ where $N \le 10$

$p_1 + p_2 = N$	2	3	5	7	p_1
2	2+2=4				
3		3+3=6	5+3=8	7+3=10	
5		3+5=8	5+5=10	7+5=12	
7		3+7=10	5+7=12	7+7=14	
p_2					

$x_i = \{2, 3, 5, \cdots, p_i, \cdots, N\} \subset \{1,2,3,\cdots,N\}$, $|x_i| \le N = 10$

By substitution $a_ix_i = y_i$,
we can obtain $y_1 + y_2 + \cdots + y_i + \cdots + y_m = N$.

Solution set $y_i$is $a_ix_i = \{a_i \cdot 1, a_i \cdot 2, \ldots, a_i \cdot k, \ldots, N\}$, therefore, $a_i \cdot k \le N$, and the number of the solution set is $|a_ix_i| = \frac{N}{a_i}$.

In the equation $x_1 + 2x_2 = x_3$, the solution sets x_1,$2x_2$ and x_3 are

$$x_1 = x_3 = \{1,2,3,\cdots,k,\cdots,N\}$$
$$2x_2 = \{2, 4, \cdots, 2k, \cdots, N\}$$

The number of the solution set $|x_1| = |x_3| = N, |2x_2| = \frac{N}{2}$.

The number of integer solutions is $NS \sim \frac{|x_1|\cdot|2x_2|\cdot|x_3|}{2!N} = \frac{N\cdot\frac{N}{2}\cdot N}{2N} = \frac{N^2}{4}$.

Density Transformation Method

We now discuss the density transformation of linear Diophantine equations whose solution sets are integer subsets.

Let a_1,a_2,...,a_m be positive integers to satisfy $gcd(a_1, a_2, \cdots, a_m) = 1$. In addition, let $NS = N(a_1,\ a_2, \cdots, a_m, N)$ be the number of integer solutions of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_mx_m = N \quad (25)$$

For non-negative integers x_1,x_2,...,x_m, according to Schur's theorem, the approximate number of integer solutions of equation (25) is

$$NS = N(a_1, a_2, \cdots, a_m; N) = \frac{N^{m-1}}{(m-1)!\cdot a_1\cdot a_2\cdot\cdots\cdot a_m} + O(N^{m-2}) \quad (26)$$

For non-negative integers$N$, provided that N is sufficiently large, especially when there exists an integerN_1such that each $N \ge N_1$has at least one representation [10].

Table 4. Total number of integer solutions of the equation $x_1 + 2x_2 = x_3$, where $x_{1,\ 3} \le 10, 2x_2 \le 10$

$x_1 + 2x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
2	1+2=3	2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=10	9+2=11		
4	1+4=5	2+4=6	3+4=7	4+4=8	5+4=9	6+4=10	7+4=11	8+4=12	9+4=13		
6	1+6=7	2+6=8	3+6=9	4+6=10	5+6=11	6+6=12	7+6=13	8+6=14	9+6=15		
8	1+8=9	2+8=10	3+8=11	4+8=12	5+8=13	6+8=14	7+8=15	8+8=16	9+8=17		
10											
$2x_2$											

$x_i \subseteq \{1,2,3,\cdots,N\}$, $x_i \ne x_j, |x_i| \ne |x_j|, |x_i| \le N = 10$

Equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$ (27)

Non-negative integers x_1,x_2, ...,x_m, and the solution sets are integer subsets.

$x_i = \{(b_i)_1, (b_i)_2, (b_i)_3, \cdots, N\} \subseteq \{1,2,3,\cdots,N\}(1 \le i \le m)$, and can be$x_i \ne x_j$. When $(b_i)_k \le N$, and the density of the solution set is $d_i = \frac{|x_i|}{N}$, and the number of the solution set is $|x_i| = d_i \cdot N \le N$.

Transform $x_i = \frac{1}{d_i} \cdot y_i$, the solution set is $y_i = \{1,2,3,\cdots,N\}(1 \le i \le m)$ is integer set.

Solution set $\frac{1}{d_i} \cdot y_i = \left\{\frac{1}{d_i} \cdot 1,\ \frac{1}{d_i} \cdot 2,\ \ldots,\ \frac{1}{d_i} \cdot k,\ \ldots,\ N\right\} \subseteq \{1, 2, 3, \cdots, k, \cdots, N\}$, and the number of the solution set is $\left|\frac{1}{d_i} \cdot y_i\right| = d_i \cdot N \le N$, the solution set is $\frac{1}{d_i} \cdot y_i$, the density is d_i, and there are

$$\frac{1}{d_1} \cdot y_1 + \frac{1}{d_2} \cdot y_2 + \cdots + \frac{1}{d_i} \cdot y_i + \cdots + \frac{1}{d_m} \cdot y_m = N \tag{28}$$

According to formula (26), the approximate number of integer solutions of equation (28) is

$$N\left(\frac{1}{d_1}, \frac{1}{d_2}, \cdots, \frac{1}{d_m}; N\right) = \frac{m}{N} \frac{1}{(m)!}(d_1 \cdot N)(d_2 \cdot N) \cdot \ldots \cdot (d_m \cdot N) + O(N^{m-2}) \tag{29}$$

where $d_i \cdot N = |x_i|$. The approximate number of integer solutions of equation (28) is also

$$NS = \frac{|x_1| \cdot |x_2| \cdots |x_m|}{N \cdot (m-1)!} + O(N^{m-2}) \tag{30}$$

Lemma 3 is proved.

For further discussion on the cases of unequal density$d_i \cdot N = |x_i|$and applications, see reference [9, 11, 12].

Lemma 4 The number of integer solutions of the equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N$ is equal to the number of integer solutions of the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof

Combinatorial approximation method

In the equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = N$, (31)

By substituting $x_1 = y_1$, $\pm x_i = y_i (2 \le i \le m)$

we can obtain $y_1 + y_2 + \cdots + y_i + \cdots + y_m = N$. (32)

Whenx_i, $y_i (1 \le i \le m) \subset \{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i \subset \{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N \le -k \le -1$, the number of the solution set is$|x_1| = |y_1|$, $|-x_i| = |x_i| = |y_i| (2 \le i \le m)$.
According to lemma 3, the approximate number of integer solutions of equation (31) can be expressed as

$$NS \sim c\frac{|x_1| \cdot |\pm x_2| \cdots |\pm x_m|}{N \cdot (m-1)!} \sim c\frac{|x_1| \cdot |x_2| \cdots |x_m|}{N \cdot (m-1)!} + O(N^{m-2}) \tag{33}$$

According to lemma 3, the approximate number of integer solutions of equation (32) can be expressed as

$$NS \sim c\frac{|y_1| \cdot |y_2| \cdots |y_m|}{N \cdot (m-1)!} \sim c\frac{|x_1| \cdot |x_2| \cdots |x_m|}{N \cdot (m-1)!} + O(N^{m-2}) \tag{34}$$

Whenx_i, $y_i (1 \le i \le m) = \{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i = \{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N \le -k \le -1$, the number of the solution set is $|x_1| = |y_1| = N$, $|-x_i| = |x_i| = |y_i| = N (2 \le i \le m)$.

According to lemma 2, the approximate number of integer solutions of equation (31) can be expressed as

$$NS \sim c\frac{|x_1| \cdot |\pm x_2| \cdots |\pm x_m|}{N \cdot (m-1)!} \sim c\frac{|x_1| \cdot |x_2| \cdots |x_m|}{N \cdot (m-1)!} = c\frac{N^{m-1}}{(m-1)!} + O(N^{m-2}) \tag{35}$$

According to lemma 2, the approximate number of integer solutions of equation (32) can be expressed as

$$NS \sim c\frac{|y_1| \cdot |y_2| \cdots |y_m|}{N \cdot (m-1)!} \sim c\frac{|x_1| \cdot |x_2| \cdots |x_m|}{N \cdot (m-1)!} = c\frac{N^{m-1}}{(m-1)!} + O(N^{m-2}) \tag{36}$$

The number of integer solutions of equation (31) is equal to the number of integer solutions of equation (32).

Proof of Lemma 4 is completed.

Lemma 5 The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ is equal to the number of integer solutions of equation$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof

Combinatorial approximation method

Equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ (37)
Becomes $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m + N = N$ (38)

By substitution$x_1 + N = y_1$and $\pm x_i = y_i (2 \le i \le m)$
we can obtain $y_1 + y_2 + \cdots + y_i + \cdots + y_m = N$. (39)

Since N is a constant; therefore, it is not included the number of the solution set. The number of solution set $|x_1 + N|$ is only regarded as $|x_1 + N|$=$|x_1| = |y_1|$.

When x_i, $y_i (1 \le i \le m) \subset \{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i = \{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N \le -k \le -1$, the number of the solution set is $|-x_i| = |x_i| = |y_i| (2 \le i \le m)$.

According to lemma 3, the approximate number of integer solutions of equation (37) can be expressed as

$$NS \sim c\frac{|x_1 + N| \cdot |\pm x_2| \cdots |\pm x_m|}{N \cdot (m-1)!} = c\frac{|x_1| \cdot |x_2| \cdots |x_m|}{N \cdot (m-1)!} + O(N^{m-2}) \tag{40}$$

According to lemma 3, the approximate number of integer solutions of equation (39) can be expressed as

$$NS \sim c\frac{|y_1| \cdot |y_2| \cdots |y_m|}{N \cdot (m-1)!} = c\frac{|x_1| \cdot |x_2| \cdots |x_m|}{N \cdot (m-1)!} + O(N^{m-2}) \tag{41}$$

Since N is a constant; therefore, it is not included the number of the solution set. The number of solution set $|x_1 + N|$ is only regarded as $|x_1 + N|$=$|x_1| = |y_1| = N$.

When x_i, $y_i(1 \le i \le m) = \{1,2,3,\cdots,k,\cdots,N\}$, the solution set of negative integers is $-x_i = \{-1,-2,-3,\cdots,-k,\cdots,-N\}$, and when $-N \le -k \le -1$, the number of the solution set is $|-x_i| = |x_i| = |y_i| = N(2 \le i \le m)$.

According to lemma 2, the approximate number of integer solutions of equation (37) can be expressed as

$$NS \sim c\frac{|x_1+N|\cdot|\pm x_2|\cdots|\pm x_m|}{N\cdot(m-1)!} \sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} = c\frac{N^{m-1}}{(m-1)!} + O(N^{m-2}) \quad (42)$$

According to lemma 2, the approximate number of integer solutions of equation (39) can be expressed as

$$NS \sim c\frac{|y_1|\cdot|y_2|\cdots|y_m|}{N\cdot(m-1)!} \sim c\frac{|x_1|\cdot|x_2|\cdots|x_m|}{N\cdot(m-1)!} = c\frac{N^{m-1}}{(m-1)!} + O(N^{m-2}) \quad (43)$$

The number of integer solutions of equation $x_1 \pm x_2 \pm \cdots \pm x_i \pm \cdots \pm x_m = 0$ is equal to the number of integer solutions of equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$.

Proof of Lemma 5 is completed.

Consider the non-uniform distribution of the (prime) number of integer solutions caused by the combination of density d_i in each m dimension with N change, and then $\frac{1}{(m-1)!}$ is modified to $c(m,d_i)$. $c(m,d_i)$ can be obtained through calculation based on precision; that is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) \sim \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| = c(m,d_i)\frac{1}{N}|x_1||x_2|\cdots|x_m| \quad (44)$$

When $m = 2$, the equation $x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$change into $x_1 + x_2 = N$. The approximate number of integer solutions to the equation $x_1 + x_2 = N$ is

$$NS = c\frac{|x_1|\cdot|x_2|}{N} + O(1)(N \to \infty) \quad (45)$$

where the solution sets are $x_i\ (i = 1,2) \subseteq \{1,2,\ 3,\cdots,j,\cdots,N\}$, and the number of solution sets is $|x_i|(i = 1,2) \le N$.

Enumeration Method of Lemma 4 and Lemma 5

Table 5. Total number of integer solutions of the 3D equation $x_1 - x_2 = x_3$ where $x_i \le 10$

$x_1 - x_2 = x_3$	1	2	3	4	5	6	7	8	9	10	x_1
1	1-1=0	2-1=1	3-1=2	4-1=3	5-1=4	6-1=5	7-1=6	8-1=7	9-1=8		
2		2-2=0	3-2=1	4-2=2	5-2=3	6-2=4	7-2=5	8-2=6	9-2=7		
3			3-3=0	4-3=1	5-3=2	6-3=3	7-3=4	8-3=5	9-3=6		
4				4-4=0	5-4=1	6-4=2	7-4=3	8-4=4	9-4=5		
5					5-5=0	6-5=1	7-5=2	8-5=3	9-5=4		
6						6-6=0	7-6=1	8-6=2	9-6=3		
7							7-7=0	8-7=1	9-7=2		
8								8-8=0	9-8=1		
9									9-9=0		
10											
x_2											

For globe
for $x_1 - x_2 = N$ $(N \ge k \ge 0)$, see total number in Table 5.
for $x_1 + x_2 = N$ $(0 \le k \le N)$, see total number in Table 1.
For local
For $x_1 - x_2 = k$, see diagonal at k in Table 5.
For $x_1 + x_2 = N - k$, see diagonal at $N - k$ in Table 1.
For other applications and validations of the approximate combinatorial method, see reference [9, 11, 12].

Proof of Theorem

Recall Lemma 3.

Lemma 3 Let $NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right)$ be the number of integer solutions of the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_i + \cdots + x_m = N$$

then the approximate number of integer solutions to the equation is

$$NS = N\left(\underbrace{1,1,\cdots,1}_{m};N\right) = \frac{1}{(m-1)\,!}\frac{1}{N}|x_1||x_2|\cdots|x_m| + O(N^{m-2})$$

For any non-negative integer N, the solution set is $x_i \subseteq \{1,2,3,\cdots,N\}$, the solution set can be $x_i \ne x_j$, and the number of the solution set is $|x_i| \le N$, it can be $|x_i| \ne |x_j|$.

Validation of Method

Chen's Theorem

In 1966, Chen Jingrun, using the sieve method, proved that sufficiently large even integers can be represented as the sum of a prime and the product of at most two primes $N = p_1 + p_2 \cdot p_3$, where p_1, p_2, p_3 are primes. The asymptotic formula for the number of such representations is: for sufficiently large even $N > 2$,

$$r_{1+2}(N) \sim \mathfrak{S}(N)\frac{2N}{(logN)^2},$$

Where $\mathfrak{S}(N) = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p>2,p|N}\frac{p-1}{p-2}$ (46)

We estimate the number of prime solutions NS to equations $p_1 + p_2 \cdot p_3 = N$ containing prime numbers, where p_1,p_2 and p_3 are prime numbers, N is integer.

The solution set p_1,p_2 and p_3 in the equation $p_1 + p_2 \cdot p_3 = N$ are prime numbers, and the number $|p_1|$ of solution set $p_1$is $|p_1| \sim \frac{N}{ln(N)}$.

The $p_2 \cdot p_3$ is called almost primenumber.

Let $\pi_k(x)$ be the number of square-free almostprime numbers$p_1 \cdot p_2 \cdots p_k \le x$, Then

$$\pi_k(x) \sim \frac{x(loglog\, x)^{k-1}}{(k-1)!\, log(x)}. \quad (47)$$

When $k = 2$, the number $|p_2 \cdot p_3|$ of solution set $p_2 \cdot p_3$ is

$$|p_2 \cdot p_3| \sim lnln(N)\frac{N}{ln(N)}. \quad (48)$$

The number of solutions NS to the equation $p_1 + p_2 \cdot p_3 = N$ is

$$NS \sim c\frac{|p_1|\cdot|p_2 \cdot p_3|}{N} = c \cdot lnln(N)\frac{N}{ln\, N}\frac{N}{ln\, N}\frac{1}{N} = c\frac{N}{(ln\, N)^2} + O(1) \quad (49)$$

The result is equivalent to Chen's result.

The weak Goldbach conjecture

The weak Goldbach conjecture states that every odd integer $N > 5$ can be represented as the sum of three prime numbers $N = p_1 + p_2 + p_3$. In 1937, I. M. Vinogradov, using the circle method, proved that sufficiently large odd integers can be represented as the sum of three primes, and gave the asymptotic formula for the number of representations: for sufficiently large odd, the number of representations is

$$r_3(N) \sim \mathfrak{S}(N)\frac{N^2}{2(ln\, N)^3} \quad (50)$$

where $\mathfrak{S}(\mathrm{N}) = \prod_{\mathrm{p}>2, p|N}(1 - \frac{1}{(\mathrm{p}-1)^2})\prod_{\mathrm{p}>2}\left(1 + \frac{1}{(\mathrm{p}-1)^3}\right)$

Estimate the number of representations$r_3(N)$ to the formula $N = p_1 + p_2 + p_3$, where p_1, p_2 and p_3 are prime numbers, N is integer, $2 < p_1, p_2, p_3 \le N$.

Using the circle method, the weak Goldbach conjecture was estimated as following numbers.

$$r_3(N) = \sum_{2<p_1,p_2,p_3 \le N}\int_0^1 e^{2\pi i(p_1+p_2+p_3-N)t}\, dt \quad (51)$$

$r_3(N)$was obtained by Ivan Vinogradov (1937) as following

$$r_3(N) = S(N)\frac{N^2}{2(ln\, N)^3}, S(N) > 0.6. \quad (52)$$

We estimate the number of solutions NS to the equation $N = p_1 + p_2 + p_3$containing prime numbers, where p_1, $p_2$and $p_3$are prime numbers, N is integer.

The solution set p_1,$p_2$and p_3 in the equation $N = p_1 + p_2 + p_3$are prime numbers, and the number $|p_1| = |p_2| = |p_3|$of solution set p_1,p_2 and $p_3$are $|p_1| = |p_2| = |p_3| \sim \frac{N}{ln(N)}$. The number of solutions NS to the equation $N = p_1 + p_2 + p_3$ is

$$NS \sim \frac{1}{2\,!}\frac{|p_1|\cdot|p_2|\cdot|p_3|}{N} = \frac{1}{2\,!}\frac{N}{ln\, N}\frac{N}{ln\, N}\frac{N}{ln\, N}\frac{1}{N} = c\frac{N^2}{(ln\, N)^3}. \quad (53)$$

The result is equivalent to Vinogradov's result.

For other applications and validations of the approximate combinatorial method, see reference [9, 11, 12].

The equivalence of the method in this paper to the circle method

The strong Goldbach conjecture. Estimate the number of representations $r_2(N)$ to formula $2N = p_1 + p_2$, where $p_1$and p_2 are prime numbers, N is integer, $2 < p_1, p_2 \le N$.

Using the circle method, the strong Goldbach conjecture was estimated as following formula.

$$r_2(N) = \sum_{2<p_1,p_2 \le N}\int_0^1 e^{2\pi i(p_1+p_2-2N)t}\, dt \quad (54)$$

This may be $r_2(N) \sim c\frac{N}{(ln\, N)^2}$.

The Hardy-Littlewood Prime Sum Conjecture. Estimate the number of representations $r_k(N)$ to formula $N = p_1 + p_2 + \cdots + p_k$, where p_1, p_2, …, and p_k are prime numbers, N is integer, $2 < p_1, p_2, \cdots, p_k \le N$.

Using the circle method, The Hardy-Littlewood prime sum conjecture was estimated as following formula.

$$r_k(N) = \sum_{2<p_1,p_2,\cdots,p_k \le N}\int_0^1 e^{2\pi i(p_1+p_2+\cdots+p_k-N)t}\, dt \quad (55)$$

This may be $r_k(N) = c\frac{N^{k-1}}{(ln\, N)^k}$

To the circle method, the strong Goldbach conjecture was estimated as following formula.

$$r_2(N) = \sum_{p_1+p_2=N,\ p_1,\ p_2 prime} 1 \quad (56)$$

$$\int_0^1 e^{2\pi i(p_1+p_2-2N)t}\, dt = \begin{cases} 1, & p_1 + p_2 = 2N \\ 0, & p_1 + p_2 \ne 2N \end{cases} \quad (57)$$

To the circle method, the Hardy-Littlewood prime sum conjecture was estimated as following formula.

$$r_k(N) = \sum_{p_1+p_2+\cdots+p_k=N,\ p_i prime} 1 \quad (58)$$

$$\int_0^1 e^{2\pi i(p_1+p_2+\cdots+p_k-N)t}\, dt = \begin{cases} 1, & p_1 + p_2 + \cdots + p_k = N \\ 0, & p_1 + p_2 + \cdots + p_k \ne N \end{cases} \quad (59)$$

To the circle method, when the equations $p_1 + p_2 = 2N$ and $p_1 + p_2 + \cdots + p_k = N$ have solutions, count 1, then obtain the sum.

To the method in this paper, count the number of solutions to the equations $p_1 + p_2 = 2N$ and $p_1 + p_2 + \cdots + p_k = N$.

The method in this paper is equivalent to the circle method.

Proof of Theorem

Theorem Equations $p_1 + p_2 + \cdots + p_k = N$, Where N, k are integer, $k \ge 1$,$N \ge N_1 N_1$ is suitably large integer, N has the same parity as k, p_i $(1 \le i \le k)$ is prime numbers. The number of integer solutions of the equation $p_1+p_2+\cdots+p_k=N$ is

$$\frac{1}{(k-1)\,!}\frac{N^{k-1}}{(\ln N)^k}+O(N^{k-2})=c\frac{N^{k-1}}{(\ln N)^k}+O(N^{k-2})$$

The number of integer solutions of the equation $p_1+p_2+\cdots+p_k=N$ is equivalent to the asymptotic formula for the number of representations $N=p_1+p_2+\cdots+p_k$. When $k=2$, it corresponds to the strong Goldbach conjecture.

Hardy-Littlewood Prime Sum Conjecture

We estimate the number of solutions NS to the equation $p_1+p_2+\cdots+p_k=N$ containing prime numbers, where p_1,p_2,⋯, and p_k are prime numbers, N is integer.

The number $|p_k|$ of solutions set p_k is $|p_k|\sim\frac{N}{ln(N)}$.

We have the approximate number of integer solutions of the equation $p_1+p_2+\cdots+p_k=N$

$$NS=\frac{1}{(k-1)\,!}\frac{|p_1|\cdot|p_2|\cdots|p_k|}{N}=\frac{1}{(k-1)\,!}\overbrace{\frac{N}{ln\,N}\cdots\frac{N}{ln\,N}}^{k}\frac{1}{N}=\frac{1}{(k-1)\,!}\frac{N^{k-1}}{(ln\,N)^k}+O(N^{k-2}) \tag{60}$$

The formula above is >1, the Hardy-Littlewood Prime Sum equation has solutions. The Hardy-Littlewood Prime Sum Conjecture holds.

Consider the non-uniform distribution of the number of integer solutions caused by the combination of density d_i in each m dimension with N change, and then $\frac{1}{(m-1)!}$ is modified to $c(m,d_i)$. $c(m,d_i)$ can be obtained through calculation based on precision

$$\frac{1}{(k-1)\,!}\frac{N^{k-1}}{(\ln N)^k}=c(m,d_i)\frac{N^{k-1}}{(ln\,N)^k}+O(N^{k-2}) \tag{61}$$

The asymptotic formula for the number of representations $N=p_1+p_2+\cdots+p_k$, or the approximate number of integer solutions of the Hardy-Littlewood Prime Sum equation $p_1+p_2+\cdots+p_k=N$

$$c(m,d_i)\frac{N^{k-1}}{(ln\,N)^k}=c_k\frac{N^{k-1}}{(ln\,N)^k}+O(N^{k-2}) \tag{62}$$

Calculating the coefficient c_k

In this paper, the density of prime number $d_i=\frac{1}{ln(N)}$, the 1 −dimensiondistribution of prime number is $\frac{N}{ln(N)}$, the k-dimension distribution of prime number is $\frac{N^{k-1}}{(ln\,N)^k}$ in here.The distribution is different between interval $[0,N_1]$ and interval $[N_1,2N_1]$, we have

$$c_k=\frac{\frac{N^{k-1}}{(ln\,N)^k}\Big|_{N=0}^{N=N_1}}{\frac{N^{k-1}}{(ln\,N)^k}\Big|_{N=0}^{N=N_1}+\frac{N^{k-1}}{(ln\,N)^k}\Big|_{N=N_1}^{N=2N_1}} \tag{63}$$

For the sake of calculation, transform formula into

$$c_k=\frac{N}{N}\frac{\frac{N^{k-1}}{(ln\,N)^k}\Big|_{N=0}^{N=N_1}}{\frac{N^{k-1}}{(ln\,N)^k}\Big|_{N=0}^{N=N_1}+\frac{N^{k-1}}{(ln\,N)^k}\Big|_{N=N_1}^{N=2N_1}}=\frac{\frac{N^k}{(ln\,N)^k}\Big|_{N=0}^{N=N_1}}{\frac{N^k}{(ln\,N)^k}\Big|_{N=0}^{N=N_1}+\frac{N^k}{(ln\,N)^k}\Big|_{N=N_1}^{N=2N_1}}=\frac{\left(\frac{N}{ln\,N}\right)^k\Big|_{N=0}^{N=N_1}}{\left(\frac{N}{ln\,N}\right)^k\Big|_{N=0}^{N=N_1}+\left(\frac{N}{ln\,N}\right)^k\Big|_{N=N_1}^{N=2N_1}}$$

$$c_k=\frac{\left(\frac{N}{ln\,N}\right)^k\Big|_{N=0}^{N=N_1}}{\left(\frac{N}{ln\,N}\right)^k\Big|_{N=0}^{N=N_1}+\left(\frac{N}{ln\,N}\right)^k\Big|_{N=N_1}^{N=2N_1}}=\frac{(\pi(N))^k\Big|_{N=0}^{N=N_1}}{(\pi(N))^k\Big|_{N=0}^{N=N_1}+(\pi(N))^k\Big|_{N=N_1}^{N=2N_1}} \tag{64}$$

Strong Goldbach Conjecture

We estimate the number of solutions$NS$to the equations $p_1+p_2=2N$, where$p_1$and p_2 are prime numbers,$N$is integer.

The solution set $p_1$and $p_2$in the equations $p_1+p_2=2N$ are prime numbers, and the number $|p_1|=|p_2|$of solution set $p_1$and p_2 are $|p_1|=|p_2|=\frac{N}{ln(N)}$. The approximate number of prime solutions to the equation $p_1+p_2=2N$ is

$$NS\sim c\frac{|p_1|\cdot|p_2|}{N}=c\frac{N}{ln\,N}\frac{N}{ln\,N}\frac{1}{N}=c\frac{N}{(ln\,N)^2}+O(1) \tag{65}$$

The formula above is >1, the strong Goldbach equationhas solutions. The strong Goldbach conjecture holds.

Consider the non-uniform distribution of the number of integer solutions caused by the combination of density d_i in each m dimension with N change, and then $\frac{1}{(m-1)!}$ is modified to $c(m,d_i)$. $c(m,d_i)$ can be obtained through calculation based on precision.

The asymptotic formula for the number of representations $p_1+p_2=2N$, or the approximate number of prime solutions of the Goldbach equation $p_1+p_2=2N$ is

$$c_2\frac{N}{(ln\,N)^2}+O(1) \tag{66}$$

Calculating the coefficient c_2

From (64), when$k=2$, c_2 is

$$c_2=\frac{(\pi(N))^2\Big|_{N=0}^{N=N_1}}{(\pi(N))^2\Big|_{N=0}^{N=N_1}+(\pi(N))^2\Big|_{N=N_1}^{N=2N_1}} \tag{67}$$

Based data in Table 6, the coefficient c_2 is computed,c_2 isc_2~0.6.

Table 6. Calculation the coefficient c_2

	N_1	$2N_1$	N_1	$2N_1$	N_1	$2N_1$
N	100	200	1000	2000	10000	20000
π(N)	25	46	168	303	1229	2262
c_2	0.586		0.607		0.586	

Proof completed.

Nevanlinna theory deals with the growth of zeros and poles (roots or solution sets) of meromorphic functions. According to Vojta's conjecture, there is a corresponding (similar) relationship between Nevanlinna theory in function fields and the abc conjecture in number fields (including integers). Since the general Fermat conjecture can be derived from the abc conjecture and concerns the number of zeros (roots or solutions) of equations, Nevanlinna theory, the abc conjecture, Vojta's conjecture, and the general Fermat conjecture all those problems is the same class of problems related to the number of zeros (roots or solutions) of equations.

This article presents an approximate combinatorial method for estimating the number of integer solutions to general Diophantine equations with countable solution sets. The applicable countable solution sets include integers, rational numbers, prime numbers, algebraic numbers, polynomials, algebraic number fields, and function fields. Some problems can also be transformed into counting number of solutions of equations. For applications of this method in other areas, such as the essence and number of solutions of abc conjecture, and so forth, please refer to the reference [9].

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