**International Journal of Science Academic Research**
Vol. 02, Issue 02, pp.1071-1075, February, 2021
Available online at http://www. scienceijsar. com
ISSN: 2582-6425

**Research Article**

# METHOD OF LOBACHEVSKI FOR SOLVING NON LINEAR ALGEBRAIC EQUATIONS WITH REAL COEFFICIENTS

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**Received** 18th December 2020; **Accepted** 13th January 2021; **Published online** 15th February 2021

**Abstract**

The method of Lobachevski permits to determine both real and complex roots of nonlinear algebraic equations with real coefficients. Its advantages are that it is not necessary to priory search the intervals of appurtenance of the roots, it simultaneously gives all the roots, reducing the computational time when using other existing iterative methods, at last, its reliability during operational works is easy.

**Keywords:** Method of Lobachevski, Nonlinear algebraic equations with real coefficients, Squaring process, Intervals of location of the roots, Method of separation of the roots.

## INTRODUCTION

Numerical treatments of empirical data usually lead to the resolution of nonlinear algebraic equations of the form

$$a_n x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0, \quad (1)$$

where $a_1$, i =0, 1, 2, …, n are real coefficients not all zeros. n - the degree of the equation.

Finding roots of such equations is one of the oldest problems of algebra. Algorithms for n = 1 and n = 2 have been elaborated. For n ≥ 3 things begin to be complicated and only numerical methods can be very helpful. Between others, the methods of tangents and secants are the most encountered, (Burden and Faires, 2005; Melentev, 1962; Patel, 1994). Their common particularity is that they permit to find only one solution, meaning that the method must be repeated n times for the n roots. An important problem to be priory solved is the finding of the interval in which the root belongs and this is not always an easy task. This article exposes a method elaborated by Lobachevski in 1834, ameliorated by Graeffe in 1837 and extended for finding complex roots by Hanke in 1841, (Lobachevsky, 1948). Its actual form was given by Krylov. It is also called the method of separation of the roots. It offers many advantages, between others, the unnecessary finding of the intervals of appurtenance of roots, the simultaneously finding of all the roots, both real and complex ones and the shorter time of execution. Unfortunately, the method of Lobachevski seems to be neglected today. With the development of science, more complicated equations of form (1) are frequently encountered and their roots could only be found if using the method of Lobachevski, whence the importance of this paper. This work has five sections. The first one is this introduction to the problem to be solved. The second one exposes the principle of the methods of Lobachevski, one for real and distinct roots and another for both real and complex roots. Implementations of both methods are given in section three and the conclusion in section four. The references in alphabetic order are in section five.

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## PRINCIPLE OF THE METHOD

Consider a nonlinear algebraic equation of power n, n ≥3:

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots\dots + a_{n-1} x + a_n = 0, \quad (2)$$

with all nonzero real coefficients $a_i$, i = [0,n].

Let $x_1, x_2, \dots, x_n$ be the n roots of (2) with:

$$|x_1|>|x_2|> \text{---} >|x_n|, \quad (3)$$

where $|x_{i\in[1,n]}|$ is the module of $x_i$.

Equation (2) can be repeatedly transformed to new nonlinear algebraic equations such that their roots are the squares of the corresponding roots of the preceding equation. Thus, if 3 and 4 were roots of (2), then after the first transformation the roots of the new equation should be $3^2$ and $4^2$ and after k transformations, $3^m$ and $4^m$ with $m = 2^k$. The ratios of the roots of the initial and final equations are respectively ¾ = 0.75 and $(3/4)^m$. For instance, if k = 5, m = 32, the second ratio should be $10^{-4}$ times smaller than the initial one. Thus, when k increases, the roots with a lowest module should be neglected compared to the one with a highest module. This is the principle of separation of the roots of an equation.

These transformations are made as follows.

Putting $x_i$, i = 1, ..., n, the roots of (2). We may also write:

$$f(x) = a_0 (x-x_1)(x-x_2)\dots(x-x_n). \quad (4)$$

From (2-3) we deduce f(-x):

$$f(-x) = a_0(-x-x_1)(-x-x_2)\dots(-x-x_n)$$
$$= (-1)^n a_0 (x+x_1)(x+x_2)\dots(x+x_n). \quad (5)$$

Multiplying (4) and (5) gives:

$$f(x)f(-x) = (-1)^n a_0^2 (x^2-x_1^2)(x^2-x_2^2)\dots(x^2-x_n^2) \quad (6)$$

Putting y = -x² in (5) gives equation φ(y) with:

$$\varphi(y) = (-1)^n a_0^2(-y-x_1^2)(-y-x_2^2)\dots(-y-x_n^2)$$
$$= a_0^2(y+x_1^2)(y+x_2^2)\dots(y+x_n^2) =$$
$$= a_0^{(1)} y^n + a_1^{(1)} y^{n-1} + \dots + a_{n-1}^{(1)} y + a_n^{(1)} = 0. \quad (7)$$

The roots of (7) are $-x_1^2, -x_2^2, \ldots, -x_n^2$.
(5) can be given the form:

$$f(-x) = a_0x^n(-1)^n + a_1x^{n-1}(-1)^{n-1} + \ldots + a_n = = (-1)^n[a_0x^n - a_1x^{n-1} + \ldots + a_n(-1)^n]. \quad (8)$$

Taking (8) into consideration, (6) becomes:

$$f(x)f(-x) = (-1)^n[a_0^2x^{2n} - (a_1^2-2a_0a_2)x^{2n-2} + (a_2^2-2a_1a_3 + 2a_0a_4)x^{2n-4} + \ldots + (-1)^n a_n^2]$$
$$= (-1)^na_0^2x^{2n} + (-1)^{n-1} (a_1^2-2a_0a_2)x^{2(n-1)} + + (-1)^{n-2} (a_2^2-2a_1a_3 + 2a_0a_4)x^{2(n-2)} + \ldots + a_n^2 = 0. \quad (9)$$

Consequently, (7) becomes:

$$\varphi(y) = a_0^2y^n + (a_1^2 - 2a_0a_2)y^{n-1} + (a_2^2 - 2a_1a_3 + 2a_0a_4)y^{n-1} + + \ldots + a_n^2 = 0. \quad (10)$$

(7) and (10) give the relationships between former and new coefficients:

$$\begin{aligned} a_0^{(1)} &= a_0^2, \\ a_1^{(1)} &= a_1^2 - 2a_0a_2, \\ a_2^{(1)} &= a_2^2 - 2a_1a_3 + 2a_0a_4, \\ &\ldots\ldots\ldots\ldots, \\ a_n^{(1)} &= a_n^2. \end{aligned} \quad (11)$$

(11) is the algorithm of computation of new coefficients $a_i^{(1)}$, i $\in$ [0,n], after the first transformation. This algorithm could be generalized for any transformation. Thus, if we are at the k-th transformation, (11) take the next form:

$$\begin{aligned} a_0^{(k)} &= (a_0^{(k-1)})^2 \\ a_1^{(k)} &= (a_1^{(k-1)})^2 - 2 a_0^{(k-1)}a_2^{(k-1)} \\ a_2^{(k)} &= (a_2^{(k-1)})^2 - 2 a_1^{(k-1)}a_3^{(k-1)} + 2 a_0^{(k-1)}a_4^{(k-1)} \\ &\ldots\ldots\ldots\ldots, \\ a_n^{(k)} &= (a_n^{(k-1)})^2 \end{aligned} \quad (11a)$$

Formulas (11a) tell us that the first and last coefficients of the transformed equation are just the squares of the corresponding ones from the preceding equation, i.e.

$a_0^{(k)} = (a_0^{(k-1)})2$ and $a_n^{(k)} = (a_n^{(k-1)})^2$

and the second and before last coefficients are obtained subtracting from the squares of their corresponding coefficients the double products of their neighbouring coefficients, i.e.

$a_1^{(k)} = (a_1^{(k-1)})^2 -2\, a_0^{(k-1)}a_2^{(k-1)}$, and $a_{n-1}^{(k)} = (a_{n-1}^{(k-1)})^2 -2\, a_{n-2}^{(k-1)}a_n^{(k-1)}$. The remaining coefficients are calculated subtracting from the squares of their corresponding coefficients the double products of their closest backward and forward coefficients and adding the double products of their own closest backward and forward coefficients, i.e.

$$a_i^{(k)} = (a_i^{(k-1)})^2 -2\, a_{i-1}^{(k-1)}a_{i+1}^{(k-1)} + 2\, a_{i-2}^{(k-1)}a_{i+2}^{(k-1)},\ i = 2, ..., n-2.$$

These computations are repeated until all the double products are negligible as null, indicating that the precision has been reached. To proceed easily and faster while avoiding errors at the same time, computations should be done according to the scheme in Table 1. To reduce round-off errors, computations should be done with at least two more significant digits above the given precision.

**Table 1. Intermediary computations of transformed coefficients**

| k | $a_0^{(k)}$ | $a_1^{(k)}$ | $a_2^{(k)}$ | .......... | $a_n^{(k)}$ |
|---|---|---|---|---|---|
| 0 | $a_0$ | $a_1$ | $a_2$ | ........... | $a_n$ |
| | $a_0^2$ | $a_1^2$ $- 2a_0a_2$ | $a_0^2$ $- 2a_1a_3$ $+2a_0a_4$ | | $a_n^2$ |
| 1 | $a_0^{(1)}$ | $a_2^{(1)}$ | $a_2^{(1)}$ | | $a_n^{(1)}$ |
| .......... | ............ | ............ | ........... | .......... | ........... |

When finding the roots of (2), the two more encountered cases are: **a)** all the roots are different real numbers; **b)** some of them are complex numbers. Let us examine each case.

**a) All the roots are real and different**.

At the k-th transformation, recall that $m=2^k$, the next system of relationships between the roots and coefficients of the given equation based on the theorem of Vieta can be established:

$$\begin{aligned} \frac{a_1^{(k)}}{a_0^{(k)}} &= x_1^m + x_2^m + \ldots + x_n^m \\ &= x_1^m\left[1 + \left(\frac{x_2}{x_1}\right)^m + \cdots + \left(\frac{x_n}{x_1}\right)^m\right], \\ \frac{a_2^{(k)}}{a_0^{(k)}} &= x_1^mx_2^m + x_1^mx_3^m + \ldots + x_{n-1}^mx_n^m \\ &= x_1^mx_2^m\left[1 + \left(\frac{x_3}{x_2}\right)^m + \cdots + \left(\frac{x_{n-1}x_n}{x_1x_2}\right)^m\right], \\ \frac{a_3^{(k)}}{a_0^{(k)}} &= x_1^mx_2^mx_3^m + x_1^mx_2^mx_4^m + \ldots + x_{n-2}^mx_{n-1}^mx_n^m \\ &= x_1^mx_2^mx_3^m\left[1 + \left(\frac{x_4}{x_3}\right)^m + \cdots + \left(\frac{x_{n-2}x_{n-1}x_n}{x_1x_2x_3}\right)^m\right], \\ &\ldots\ldots\ldots\ldots, \\ \frac{a_n^{(k)}}{a_0^{(k)}} &= x_1^mx_2^mx_3^m \ldots x_n^m, \end{aligned} \quad (12)$$

Based on (3) and on the fact that m is generally high, the following equalities can be deduced:

$$\begin{aligned} x_1^m &= \frac{a_1^{(k)}}{a_0^{(k)}}, \\ x_1^mx_2^m &= \frac{a_2^{(k)}}{a_0^{(k)}}, \\ x_1^mx_2^mx_3^m &= \frac{a_3^{(k)}}{a_0^{(k)}}, \\ &\ldots\ldots\ldots \\ x_1^mx_2^mx_3^m \ldots\ldots\ x_n^m &= \frac{a_n^{(k)}}{a_0^{(k)}}, \end{aligned} \quad (13a)$$

whence:

$$\begin{aligned} x_1^m &= \frac{a_1^{(k)}}{a_0^{(k)}}, \\ x_2^m &= \frac{a_2^{(k)}}{a_1^{(k)}}, \\ x_3^m &= \frac{a_3^{(k)}}{a_2^{(k)}}, \\ &\ldots\ldots\ldots \\ x_n^m &= \frac{a_n^{(k)}}{a_{n-1}^{(k)}}, \end{aligned} \quad (13b)$$

System (13b) gives at all the searched roots of (2) affected to power m. As m is an even number, $x_i = \pm\sqrt{x_i^m}$, i = 1, 2, .., n. Substituting each value in (2) enables us to determine which sign should be considered.

**b) Some roots are conjugate complex numbers**

Let us divide both members of (2) by $a_0 \neq 0$. We obtain the next expression called the monic form of (2) as $b_0 = 1$:

$$x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n = 0. \quad (14)$$

If (14) has only a couple of conjugate complex roots, say $x_2$ and $x_3$, so the remaining n-2 roots are real numbers which can be noted at the k-th transformation by $\beta_i = x_i^m$, i = 1, 2, ..., n-2, the two complex ones been $x_2^m$ and $x_2^m$. So based on (13a), we may write the next relations:

$$b_1 = \beta_1,\ b_2 = \beta_1\beta_2,\ b_3 = \beta_1\beta_2\beta_3,\ ...,\ b_p = \beta_1\beta_2\beta_3...\ \beta_p,\ p = n\text{-}2, \quad (15a)$$

for the n-2 real roots and:

$$x_2^m = \rho_1(\cos\psi_1 + i\sin\psi_1),\ x_2^m = \rho_1(\cos\psi_1 - i\sin\psi_1), \quad (15b)$$

for the two complex roots.

From (15a), we have:

$$\beta_1 = b_1,\ \beta_2 = \frac{b_2}{b_1},\ \beta_3 = \frac{b_3}{b_2},\ ...,\ \beta_p = \frac{b_p}{b_{p-1}}, \quad (16a)$$

Whence the real roots given by:

$$x_i = \pm\sqrt[m]{\beta_i},\ i = 1, 2, ..., p. \quad (16b)$$

Let us find the complex roots. As $x_2$ and $x_3$ are complex, it comes that the second real root, $\beta_2$, corresponds to $x_4^m$. Thus based on (3), (12), and (15a), we may write:

$$\begin{aligned} x_1^m &= \beta_1, \\ x_2^m &= \rho_1(\cos\varphi_1 + i\sin\varphi_1), \\ x_3^m &= \rho_1(\cos\varphi_1 - i\sin\varphi_1), \\ x_4^m &= \beta_2, \\ &\ldots\ldots \end{aligned} \quad (17)$$

Recalling that:

$$\begin{aligned} &x_2^m + x_3^m = 2\rho_1\cos\varphi_1 = 2\rho_1\cos\varphi_1, \quad (a) \\ &x_2^m x_3^m = \rho_1^2 = r^{2m}, \quad (b) \end{aligned} \quad (18)$$

Taking into account (17) and (18), (15a) becomes:

$$\begin{aligned} b_1 &= \beta_1, \\ b_2 &= x_1^m x_2^m + x_1^m x_3^m = \beta_1 2\rho_1\cos\varphi_1, \\ b_3 &= x_1^m x_2^m x_1^m = \beta_1\rho_1^2, \\ b_4 &= x_1^m x_2^m x_1^m x_4^m = \beta_1\rho_1^2\beta_2, \\ &\ldots\ldots \end{aligned} \quad (19)$$

From the first and third equations of system (19), we have the module r of the complex roots:

$$\frac{b_3}{b_1} = \rho_1^2,\ r = \sqrt[m]{\rho_1^2}. \quad (20)$$

From the first and second equations of system (19), we have:

$$\frac{b_3}{b_1} = 2\rho_1\cos\varphi_1 = 2\rho_1\cos m\psi \quad (21)$$

Recalling that:

$$b_1 = -(x_1^m + 2r_1\cos\varphi_1 + x_4^m + \ldots + x_n^m), \quad (22)$$

and considering (21), the argument of the complex roots is easily found.

Suppose that (2) has more than one couple of conjugate complex roots, say two such roots. In this case, we have two conjugate complex numbers, i.e. four complex roots. We must form a system of two equations (similar to (22)) to solve for the arguments of the complex roots. Their modules are obtained using formula similar to (20).

To obtain the system of the two equations for the arguments, we proceed dividing both members of (2) by $a_0x^n \neq 0$:

$$\frac{1}{a_0} + \frac{a_{n-1}}{a_0}.\frac{1}{x} + \frac{a_{n-2}}{a_0}.\frac{1}{x^2} + \ldots + \frac{a_1}{a_0}.\frac{1}{x^{n-1}} + .\frac{1}{x^n} = 0. \quad (23)$$

Putting $\frac{1}{x} = y$, we have:

$$y^n + \frac{a_1}{a_0}y^{n-1} + \frac{a_2}{a_0}y^{n-2} + \ldots + \frac{a_{n-1}}{a_0}y + \frac{1}{a_0} = 0. \quad (24)$$

Based on the formula of Vieta we have:

$$\frac{a_1}{a_0} = -\Big(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \ldots + \frac{1}{\beta_j} + \frac{1}{r_{1(\cos\varphi_1 + i\sin\varphi_1)}} + \frac{1}{r_{1(\cos\varphi_1 - i\sin\varphi_1)}} + \frac{1}{r_{2(\cos\varphi_2 + i\sin\varphi_2)}} + \frac{1}{r_{2(\cos\varphi_2 - i\sin\varphi_2)}}\Big),\ j = 1, 2, ..., n - 4 \quad (25)$$

Recalling that:

$$\frac{1}{r\,(\cos\varphi + i\sin\varphi)} + \frac{1}{r\,(\cos\varphi - i\sin\varphi)} = \frac{2\cos\varphi}{r}. \quad (26)$$

and based on (26), expression (25) becomes:

$$\frac{a_1}{a_0} = -\Big(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \ldots + \frac{1}{\beta_j} + \frac{2\cos\varphi_1}{r_1}.+\frac{2\cos\varphi_2}{r_2}\Big) \quad (27)$$

Thus, the system equations to solve for the arguments are:

$$\begin{aligned} &a_{n-1} = -(\beta_1 + \beta_2 + \ldots + \beta_j + 2r_1\cos\varphi_1 + 2r_2\cos\varphi_2), \\ &\frac{a_1}{a_0} = -\Big(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \ldots + \frac{1}{\beta_j} + \frac{2\cos\varphi_1}{r_1}.+\frac{2\cos\varphi_2}{r_2}\Big). \end{aligned} \quad (28)$$

Knowing the cosines and modules of the two couples of conjugate complex roots, it becomes easier to find these roots solving two quadratic equations of the form $x^2 - 2r\cos\varphi x + r^2 = 0$ by the quadratic formulas.

## IMPLEMENTATION OF THE METHODS

These methods are implemented on equations obtained during practical works on the numerical methods frequently used in weather forecast at the Hydrometeorological Institute of Leningrad, State Hydrometeorological University of Saint Petersburg, today, Russia.

**Case of real and distinct roots**

Let us solve the following equation:

$$f(x) = 1.23x^5 - 2.52x^4 - 16.1x^3 + 17.3x^2 + 29.4x - 1.34 = 0 \quad (29)$$

The computations in Table 2 are stopped for k = 6, i.e. $m = 2^k = 2^6 = 64$ as at this stage, all the double products, $2a_{i-1}a_{i+1}$ and $2a_{i-2}a_{i+2}$, are at most $10^{-4}$ times lower than the main terms, $a_i^2$. Using Table 2, (13b) gives:

**Table 2.Table of intermediary computations of the roots of (3-1)**

| k | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|---|
| 0 | 1.23 | -2.52 | -16.1 | 17.3 | 29.4 | -1.34 |
| | 1.5129 | 6.3504<br>+39.6060 | 2.5921 $10^2$<br>+0.8719 $10^2$<br>+0.7232 $10^2$ | 2.9929 $10^2$<br>+9.4668 $10^2$<br>+0.0675 $10^2$ | 8.6436 $10^2$<br>+0.4633 $10^2$ | 1.7956 |
| 1 | 1.5129 | 0.4596 $10^2$ | 4.1873 $10^2$ | 1.2527 $10^3$ | 9.1072 $10^2$ | 1.7956 |
| | 2..2884 | 2.1120 $10^3$<br>-1.2670 $10^3$ | 1.7533 $10^5$<br>-1.1514 $10^5$<br>+0.0276 $10^5$ | 1.5693 $10^6$<br>-0.7627 $10^6$<br>+0.0002 $10^6$ | 8.2942 $10^5$<br>-0.0450 $10^5$ | 3.2242 |
| 2 | 2.2880 | 0.8450 $10^3$ | 0.6295 $10^5$ | 0.8068 $10^6$ | 8.2492 $10^5$ | 3.2242 |
| | 5.2391 | 7.1404 $10^5$<br>$^-$2.8815 $10^5$ | 3.9622 $10^9$<br>-1.3635 $10^9$<br>0.0040 $10^9$ | 6.5092 $10^{11}$<br>$^-$1.0385 $10^{11}$<br>+0.0000 $10^{11}$ | 6.8049 $10^{11}$<br>0.0000 $10^{11}$ | 10.3955 |
| 3 | 5.2391 | 4.2589 $10^5$ | 2.6027 $10^9$ | 5.4707 $10^{11}$ | 6.8049 $10^{11}$ | 10.3955 |
| | 2.7448 $10^1$ | 1.8138 $10^{11}$<br>-0.2727 $10^{11}$ | 6.7740 $10^{18}$<br>-0.4660 $10^{18}$<br>+0.0000 $10^{18}$ | 2.9930 $10^{23}$<br>-0.0354 $10^{23}$<br>+0.0000 $10^{23}$ | 4.6307 $10^{23}$<br>-0.0000 $10^{23}$ | 1.0807 102 |
| 4 | 2.7448 $10^1$ | 1.5411 $10^{11}$ | 6.3080 $10^{18}$ | 2.9576 $10^{23}$ | 4.6307 $10^{23}$ | 1.0807 102 |
| | 7.5340 $10^2$ | 2.3750 $10^{22}$<br>-00346 $10^{22}$ | 3.9791 $10^{37}$<br>-0.0091 $10^{37}$<br>+0.0000 $10^{37}$ | 8.7474 $10^{46}$<br>-0.0001 $10^{46}$<br>+0.0000 $10^{46}$ | 2.1443 $10^{47}$<br>-0.0000 $10^{47}$<br>-0.0000 $10^{47}$ | 1.1678 104 |
| 5 | 7.5340 $10^2$ | 2.3404 $10^{22}$ | 3.9700 $10^{37}$ | 8.7473 $10^{46}$ | 2.1443 $10^{47}$ | 1.1678 104 |
| | 0.5676 $10^6$ | 5.4773 $10^{44}$<br>-0.0001 $10^{44}$ | 1.5761 $10^{75}$<br>-0.0000 $10^{75}$<br>+0.0000 $10^{75}$ | 7.6515 $10^{93}$<br>-0.0000 $10^{93}$<br>+0.0000 $10^{93}$ | 4.5980 $10^{94}$<br>-0.0000 $10^{94}$ | 1.3638 108 |
| 6 | 0.5676 $10^6$ | 5.4774 $10^{44}$ | 1.5761 $10^{75}$ | 7.6715 $10^{93}$ | 4.5980 $10^{94}$ | 1.3638 108 |

**Table 3. Table of intermediary computations of the roots of (3-2)**

| k | $a_0^{(k)}$ | $a_1^{(k)}$ | $a_2^{(k)}$ | $a_3^{(k)}$ | $a_4^{(k)}$ |
|---|---|---|---|---|---|
| 0 | 1 | 0.68342 | 1.95562 | 0.37654 | 1.79420 |
| | 1 | 0.46706<br>- 3.91124 | 3.82445<br>- 0.51467<br>+ 3.58840 | 0.14178<br>- 7.01755 | 3.21915 |
| 1 | 1 | - 3.44418 | 6.89818 | - 6.87577 | 3.21915 |
| | 1 | 11.86238<br>- 13.79636 | 47.58489<br>- 47.36278<br>6.43830 | 47.27621<br>- 44.41255 | 10.36293 |
| 2 | 1 | - 1.93398 | 6.66041 | 2.86366 | 10.36293 |
| | 1 | 3.74028<br>- 13.32082 | 44.36106<br>11.07652<br>20.72586 | 8.20055<br>- 138.04273 | 107.39032 |
| 3 | 1 | - 9.58054 | 76.16344 | - 129.84218 | 107.39032 |
| | 1 | 91.78675<br>- 152.32688 | $5.80087 \cdot 10^3$<br>$-2.4879 \cdot 10^3$<br>$0.21478 \cdot 10^3$ | $1.68590 \cdot 10^4$<br>$-1.6358 \cdot 10^4$ | $1.15327 \cdot 10^4$ |
| 4 | 1 | $-6.05401 \cdot 10^1$ | $3.52773 \cdot 10^3$ | $0.05006 \cdot 10^4$ | $1.15327 \cdot 10^4$ |
| | 1 | $3.66511 \cdot 10^3$<br>$-7.05546 \cdot 10^3$ | $1.24449 \cdot 10^7$<br>$6.06127 \cdot 10^4$<br>$2.30654 \cdot 10^4$ | $2.50600 \cdot 10^5$<br>$-8.1368 \cdot 10^7$ | $1.33003 \cdot 10^8$ |
| 5 | 1 | $-3.39035 \cdot 10^3$ | $1.25286 \cdot 10^7$ | $-8.1118 \cdot 10^7$ | $1.33003 \cdot 10^8$ |
| | 1 | $11.49447 \cdot 10^6$<br>$-2.50572 \cdot 10^7$ | $1.56966 \cdot 10^{14}$<br>$5.50036 \cdot 10^{11}$<br>$2.66006 \cdot 10^8$ | $6.58011 \cdot 10^{15}$<br>$-3.3327 \cdot 10^{15}$ | $1.76898 \cdot 10^{16}$ |
| 6 | 1 | $-1.35627 \cdot 10^7$ | $1.56911 \cdot 10^{14}$ | $3.24743 \cdot 10^{15}$ | $1.76898 \cdot 10^{16}$ |
| | 1 | $1.83947 \cdot 10^{14}$<br>$-3.13822 \cdot 10^{14}$ | $2.46211 \cdot 10^{28}$<br>$8.80878 \cdot 10^{22}$<br>$3.53796 \cdot 10^{16}$ | $1.05458 \cdot 10^{31}$<br>$-5.5514 \cdot 10^{30}$ | $3.12929 \cdot 10^{32}$ |
| 7 | 1 | $-1.29875 \cdot 10^{14}$ | $2.46211 \cdot 10^{28}$ | $0.49944 \cdot 10^{31}$ | $3.12929 \cdot 10^{32}$ |

$x_1^{64} \approx \frac{5.4774 \cdot 10^{44}}{0.5676 \cdot 10^{6}}$, $x_2^{64} \approx \frac{1.5761 \cdot 10^{75}}{5.4774 \cdot 10^{44}}$, $x_3^{64} \approx \frac{7.6514 \cdot 10^{93}}{1.5761 \cdot 10^{75}}$,

$x_4^{64} \approx \frac{4.5980 \cdot 10^{94}}{7.5614 \cdot 10^{93}}$, $x_5^{64} \approx \frac{1.3638 \cdot 10^{8}}{4.5980 \cdot 10^{94}}$

whence:

$x_1 \approx \pm$ 4.0657; $x_2 \approx \pm$ 2.9917; $x_3 \approx \pm$ 1.9587; $x_4 \approx \pm$ 1.0284; $x_5 \approx \pm$ 0.0445.

By substitution in the initial equation we have the searched roots:

$x_1 \approx$ 4.0657; $x_2 \approx$ - 2.9917; $x_3 \approx$ 1.9587; $x_4 \approx$ - 1.0284; $x_5 \approx$ 0.0445.

**Case of real distinct and complex roots**

Let us solve the following equation in its monic form:

$$x^4 + 0.68342\, x^3 + 1.95562\, x^2 + 0.37654\, x + 1.79420 = 0 \quad (30)$$

This equation of degree four must have four roots. Intermediary computations are presented in Table 3.

The variations of coefficients $a_1^{(k)}$ and $a_3^{(k)}$ in Table 3 are not homogeneous as the double products of the corresponding coefficients remain of the same order as the square of the main coefficients. Moreover their signs are not constant when passing from iteration to another. As two columns are concerned, the initial equation has two couples of conjugate complex roots, so all the roots are complex. At k = 7, m = $2^7$ = 128, these double products do not more affect the square of the main coefficients in column $a_2^{(k)}$. This indicates that we can stop the squaring process.

Formula (20) permits us to find the modules of the complex roots:

$(r_1^2)^{128} = \frac{2.46211 \cdot 10^{28}}{1}$, $r_1 = 1.29093$,

$(r_2^2)^{128} = \frac{3.12929 \cdot 10^{32}}{2.46211 \cdot 10^{28}}$, $r_2 = 1.03760$.

From (28) we have the system of two equations to be solved for the cosines of the arguments:

$0.68342 = -(2.58186\cos\varphi_1 + 2.07520\cos\varphi_2)$,

$\frac{0.37654}{1.79420} = -\left(\frac{2\cos\varphi_1}{1.29093} + \frac{2\cos\varphi_2}{1.03760}\right)$.

We have:
$\cos\varphi_1 = -0.50063$ and $\cos\varphi_2 = 0.29354$.

Letting $x_{1,2}$ and $x_{3,4}$ the two couples of conjugate complex roots and recalling that:

$x_1 + x_2 = 2\cos\varphi_1 = -1.29259$, $x_1x_2 = r_1^2 = 1.66644$,
$x_3 + x_4 = 2\cos\varphi_2 = 0.60917$, $x_3x_4 = r_2^2 = 1.07667$,

the initial equation can be put into the product of two quadratic factors:

$x^4 + 0.68342x^3 + 1.95562x^2 + 0.37654x + 1.79420 =$
$(x^2 + 1.29259x + 1.66644)(x^2 - 0.60917x + 1.07667) = 0$,

and solving each one by quadratic formulas gives the searched complex roots.

**Conclusion**

It is obvious that the method of Lobachevski is easily applicable when searching roots of nonlinear algebraic equations. No powerful computer is needed and a pocket simple scientific calculator can be used. This method gives all the roots at once, compare to other frequently encountered iterative methods, without wasting time searching their intervals of appurtenance. Thus, the method of Lobachevski should be widely recommended for operational works.

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