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Research Article

WEAKLY FEEBLY CONTINUOUS FUNCTIONS

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Abstract

In this paper, I introduce a new class of functions weaker than that of feebly continuous functions called weakly feebly continuous functions. Several properties of this class of functions are established.

Keywords: Feebly open and feebly closed sets, Feebly continuous functions, Semi-open and semi-closed sets, Feebly connected spaces.
Subject classification: Primary-54A40, 54C10, 54D10, Secondary- 54C08.

INTRODUCTION

In 1982, Maheshwari and Jain (1) introduced the class of feebly continuous functions V. Popa (2) has further investigated the function and gave some more characterizations of feebly continuous functions. In 1997, Mustafa Cicek (3) proved that every surjective semi-continuous function is feebly continuous. Several properties of feebly continuous function and other classes of functions are obtained in (4,5,6). A function is said be semi-continuous (7) if the inverse image of every open set is semi-open.. In this paper, a weak form of feeble continuity is defined and studied. I shall see that weak feeble continuity is strictly weaker than feeble continuity. Interrelationships of weakly feebly continuous functions and other known classes of functions are obtained.

Preliminaries

The symbols X, Y and Z denote topological spaces with no separtion axioms assumed unless explicity stated. The closure and interior of a set $A \subseteq X$ are denoted by cl(A) and int(A), respectively. A subset A of X is said to be semi-open (7) if there exists an open set U such that $U \subseteq A \subseteq cl(U)$.The complement of a semi-open set is called semi-closed(8). The semi-closure (8) of a subset B of X is the intersection of all semi-closed sets containing B. It is denoted by scl(B). In 1978, Maheshwari and Jain (1) introduced the concept of feebly open sets using the notion of semi-closure. It was further investigated in (9). Feebly open sets and Feebly closed are studied further in (10),(11) and (12). A set A in X is said to be feebly open if for some open set O, $O \subseteq A \subseteq scl(O)$. It is proved that open $\Rightarrow$ feebly open $\Rightarrow$ semi-open and these implications are not reversible in general. The complement of a feebly open set is termed feebly closed (1). The intersection of all feebly closed sets containing a set A is the feeble closure of A and is denoted by Fcl(A). The feeble interior of A is also defined. It is denoted by Fint(A).

Definition 1. A function $f: X \to Y$ is said to be feebly continuous (1) if the inverse image of every open subset of Y is feebly open in X.

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Definition 2. A function $f: X \to Y$ is said to be strongly feebly continuous (1) if the inverse image of every feebly open subset of Y is feebly open in X.

Definition 3. Let X be a topological space and a point $x \in X$. A set V is called a feebly-neighbourhood of x (3) if there exists a feebly open set $U \subseteq X$ such that $x \in U \subseteq V$.

Definition 4. A space X is said to be feebly-T_2 (1) if for any pair of distinct points x and y, there exist two disjoint feebly open sets U and V such that $x \in U$ and $y \in V$.

Theorem 1. (1) A set $A \subseteq X$ is feebly open if and only if $A \subseteq scl(\text{int} A)$.

Theorem 2. (1) A set $A \subseteq X$ is feebly closed if and only if $\sin t(cl(A)) \subseteq A$.

Theorem 3. (2) For a single-valued function $f: X \to Y$, the following statements are equivalent:

1. f is feebly continuous at $x \in X$.
2. For any open set $G \subseteq Y$ such that $f(x) \in G$, it follows $x \in scl[\text{int} f^{-1}(G)]$.
3. For any semi-open set $V \subseteq X$ such that $x \in V$ and any open neighbourhood G of f(x), there exists a non-empty open set $U \subseteq V$ such that $f(U) \subseteq G$.

Theorem 4. (2) Let $f: X \to Y$ be a single-valued function. Then the following statements are equivalent:

1. f is feebly continuous.
2. The inverse image of each open set is feebly open.
3. The inverse image of each closed set is feebly closed.
4. For each subset A of X , f(Fcl(A) $\subseteq$ cl(A).
5. For each subset B of Y, Fcl($f^{-1}(B)$) $\subseteq f^{-1}(cl(B)$.
6. For each subset B of Y, $f^{-1}(cl(B)) \supseteq \sin t\, cl(f^{-1}(B))$.
7. For each subset A of X , f(sint cl(A)) $\subseteq$ cl(f(A)).

Definition 5. A function $f: X \to Y$ is said to be weakly feebly continuous if for each point $x \in X$ and for each open set V in Y containing f(x), there exists a feebly open set U in X such that $x \in U$ and $f(U) \subseteq Fcl(V)$.

Obviously, continuity ⇒ feeble continuity ⇒ weakly feeble continuity. But these implications are not reversible. Following is an example:

Example 1. Let $X = \{a, b, c\}$, $\tau_X = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $Y = \{a, b\}$, $\tau_Y = \{Y, \{a\}, \phi\}$. Let $f: X \to Y$ be defined by $f(a) = f(c) = a$ and $f(b) = b$. Then f is weakly feebly continuous but not feebly continuous.

Theorem 5. Let Y be a regular space. Then a function $f: X \to Y$ is feebly continuous if and only if it is weakly feebly continuous.

Proof: Let $x \in X$ and V be any open set containing f(x). Since Y is regular, there exists and open set O such that $f(x) \in O \subseteq cl(O) \subseteq V$. Since f is weakly feebly continuous, there exists a feebly open set U containing x such that $f(U) \subseteq Fcl(O)$. Hence $f(U) \subseteq V$.

Theorem 6. A function $f: X \to Y$ is weakly feebly continuous iff for each open set V of Y, $f^{-1}(V) \subseteq F\text{int}\left(f^{-1}(fcl(V)\right)$.

Proof. Assume that f is weakly feebly continuous. Let V be an open set in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is weakly feebly continuous, there is a feebly open set U containing x such that $f(U) \subseteq Fcl(V)$.

Thus $U \subseteq f^{-1}(Fcl(V))$ and hence $U \subseteq F\text{int}\left(f^{-1}(Fcl(V))\right)$. Thus $f^{-1}(V) \subseteq F\text{int}\left(f^{-1}(Fcl(V))\right)$. Conversely, let $x \in X$ and V be an open set containing f(x). By hypothesis, $f^{-1}(V) \subseteq F\text{int}[f^{-1}(Fcl(V))]$.

Let $U = F\text{int}[f^{-1}(Fcl(V))]$. Then $x \in U$ and U is feebly open. Further $U \subseteq f^{-1}(Fcl(V))$ or $f(U) \subseteq Fcl(V)$. Hence f is weakly feebly continuous.

Theorem 7. If $f: X \to Y$ is weakly feebly continuous, then $Fcl\left(f^{-1}(V)\right) \subseteq f^{-1}(cl(V))$ for every open subset V of Y.

Proof. Assume that there is a point $x \in Fcl\left(f^{-1}(V)\right) - f^{-1}(cl(V))$. Then $f(x) \notin cl(V)$. Hence there exists an open set O containing f(x) such that $O \cap V = \phi$ and so $cl(O) \cap V = \phi$. Since f is weakly feebly continuous, there is a feebly open set U containing x such that $f(U) \subseteq Fcl(O)$. Thus $f(U) \cap V = \phi$. Now, since $x \in Fcl\left(f^{-1}(V)\right)$ and U is a feebly-neighbourhood of x, therefore, $U \cap f^{-1}(V) \neq \phi$, which is a contradiction. Thus $Fcl\left(f^{-1}(V)\right) \subseteq f^{-1}(cl(V))$.

Theorem 8. If $f: X \to Y$ is open and weakly feebly continuous, then $f(Fcl(A)) \subseteq cl(f(A))$ for each open set A of X.

Proof. Let A be an open set in X. Let $B = f(A)$. Then $A \subseteq f^{-1}(B)$. Since f is open, B is open in Y. Since f is weakly feebly continuous, it follows that $Fcl\left(f^{-1}(B)\right) \subseteq f^{-1}(cl(B))$. Thus $Fcl(A) \subseteq f^{-1}(cl(B))$ and hence $f(Fcl(A)) \subseteq cl(B) = cl(f(A))$.

Definition 6. A space X is said to be feebly connected if it is not the union of two non-empty disjoint feebly open sets.

Theorem 9. If X is feebly connected and $f: X \to Y$ is weakly feebly continuous surjection, then Y is connected.

Proof. Assume that Y is disconnected. Then there exist non-empty open sets U_1 and U_2 such that $Y = U_1 \cup U_2$ and $U_1 \cap U_2 = \phi$. Thus $f^{-1}(U_1) \cap f^{-1}(U_2) = \phi$ and $X = f^{-1}(U_1) \cup f^{-1}(U_2)$. Since f is surjective, $f^{-1}(U_i) \neq \phi$ for $i = 1, 2$. Then by Theorem 2, $f^{-1}(U_i) \subseteq F\text{int}\left(f^{-1}(Fcl(V_i))\right)$. Since U_i is open and also closed, $f^{-1}(U_i) \subseteq F\text{int}\left(f^{-1}(U_i)\right)$. Hence $f^{-1}(U_i)$ is feebly open in X. Hence X is not feebly connected and so Y is connected.

Theorem 10. If $f: X \to Y$ is strongly feebly continuous and $g: Y \to Z$ is weakly feebly continuous, then $gof: X \to Z$ is weakly feebly continuous.

Proof. Let $x \in X$ and A be an open set in Z containing gof(x). Since g is weakly feebly continuous, there exists a feebly open set U containing f(x) such that $g(U) \subseteq Fcl(A)$. Since f is strongly feebly continuous, $f^{-1}(U)$ is feebly open in X. Let $G = f^{-1}(U)$. Then G is a feebly open set containing x such that $gof(G) \subseteq Fcl(A)$. Hence gof is weakly feebly continuous.

Lemma 1. Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of topological spaces. Then $Fcl\left(\prod_{\alpha \in \Delta} A_\alpha\right) \subseteq \prod_{\alpha \in \Delta} Fcl(A_\alpha)$, where A_α is a non-empty subset of X_α for each α.

Proof: Assume $x = (x_\alpha) \notin Fcl(\prod A_\alpha)$ and I prove that $x_\alpha \in Fcl(A_\alpha)$ for each $\alpha \in \Delta$. Let $\alpha_0 \in \Delta$. Let $X_{\alpha_0} \in B_{\alpha_0}$, where B_{α_0} is feebly open in X_{α_0} and $B = B_{\alpha_0} \times \prod_{\alpha} \{X_\alpha : \alpha \in \Delta - \{\alpha_0\}\}$. Then $x \in B$ and B is feebly open. Since $x \in Fcl(K)$ iff $K \cap M \neq \phi$ for every feebly open set M containing x (4). Therefore, $\left(B_{\alpha_0} \cap A_{\alpha_0}\right) \times \prod_{\alpha \in \Delta} A_\alpha = B \cap \left(\prod_{\alpha \in \Delta} A_\alpha\right) \neq \phi$. Then $B_{\alpha_0} \cap A_{\alpha_0} \neq \phi$ and so $x = (x_\alpha) \notin Fcl(\prod A_\alpha)$.

Theorem 11. If $f : X \to Y$ is a function and $g : X \to X \times Y$ is the graph of f, where $g(x) = (x, f(x))$. Then f is weakly feebly continuous, if g is weakly feebly continuous.

Proof: Suppose g is weakly feebly continuous. Let $x \in X$ and A be any open set containing f(x). Then $X \times A$ is an open set in $X \times Y$ containing g(x). Since g is weakly feebly continuous, there exists a feebly open set H in X containing x with

$g(H) \subseteq Fcl(X \times A) \subseteq X \times Fcl(A)$ (Lemma 1)

Now, g is the graph of f implies $f(H) \subseteq Fcl(A)$. Hence f is weakly feebly continuous.

Theorem 12. If f is weakly feebly continuous and g is a continuous function from a space X into a Hausdorff space Y, then the set $K = \{x \in X, f(x) = g(x)\}$ is feebly closed in X.

Proof: If $x \notin A$. Then $f(x) \neq g(x)$. Y is Hausdorff, implies that there exist disjoint open sets A and B such that $f(x) \in A$, $g(x) \in B$. Then $cl(A) \cap B = \phi$. Since f is weakly feebly continuous, there exists a feebly open set U containing x such that $f(U) \subseteq Fcl(A) \subseteq cl(A)$. Since g is continuous, there is an open set V containing x and $g(V) \subseteq B$. Then $U \cap V$ is a feebly open set containing x such that $(U \cap V) \cap A = \phi$ (9). Hence A is feebly closed.

Definition 7. A space X is said to be feebly-T_2 if distinct points of X have disjoint feebly-neighbourhoods.

Theorem 13. If Y is a Urysohn space and $f : X \to Y$ is weakly feebly continuous injection, then X is feebly-T_2.

Proof: Let x_1 and x_2 be two distinct points of X. then $f(x_1) \neq f(x_2)$. Since Y is Urysohn, there exist open sets U_1 and U_2 such that $f(x_1) \in U_1$, $f(x_2) \in U_2$ and $cl(U_1) \cap cl(U_2) = \phi$.

Now $F\,int\left(f^{-1}(cl(U_1))\right) \cap F\,int\left[f^{-1}(cl(U_2))\right] = \phi$. By Theorem (2),

$x_i \in f^{-1}(cl(U_i)) \subseteq F\,int\left[f^{-1}(cl(U_i))\right]$ for $i = 1, 2$.

Thus $F\,int\left[f^{-1}(cl(U_i))\right]$ and $F\,int\left[f^{-1}(cl(U_2))\right]$ are disjoint feebly neighbour hoods of x_1 and x_2, respectively. Hence X is feebly-T_2.

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