International Journal of Science Academic Research
Vol. 02, Issue 05, pp.1447-1450, May, 2021
Available online at http://www.scienceijsar.com

ISSN: 2582-6425

Research Article

FUZZY SATISFACTORY *BCK*-FILTERS

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Received 18th March 2021; **Accepted** 11th April 2021; **Published online** 24th May 2021

Abstract

The concept of satisfactory filters of *BCK*-algebras was introduced by Jun [9]. In this paper, we define the notion of fuzzy satisfactory *BCK*-filters and investigate some of its properties.

Keywords: *BCK*-algebras, filter, fuzzy filter, fuzzy satisfactory filter.

INTRODUCTION

The notion of *BCK*-algebra was introduced by Imai and Iseki in 1966 [5]. In same year, Iseki [6] introduced the notion of *BCI*-algebra which is a generalization of a *BCK*-algebra. After the introduction of the concept of fuzzy subsets by Zadeh [17], several researchers worked on the generalization of the notion fuzzy sets. Jun [9] introduced the notion of a satisfactory filter of a *BCK*-algebra and prove several basic properties which are related to satisfactory filter. In this paper, we introduce the notion of fuzzy satisfactory *BCK*-filter and prove some their fundamental properties.

Preliminaries

We review some basic definitions and properties that will be useful in our results. A *BCK-algebra* X is defined to be an algebra $(X,*,0)$ of type $(2,0)$ satisfying the following conditions

$BCK-1((xy)(xz))(zy)=0,$
$BCK-2(x(xy))y=0,$
$BCK-3xx=0,$
$BCK-4\ 0x=0,$
$BCK-5xy=0$ and $yx=0 \Rightarrow x=y,$

for all $x,y,z\in X$, where $xy=x*y$, and $xy=0$ if and only if $x\leq y$.

In a *BCK*-algebra X, the following properties hold for all $x,y,z\in X$:

P-1 $x0=x$,
P-2 $(xy)z=(xz)y$,
P-3 $x\leq y$ implies that $xz\leq yz$ and $zy\leq zx$,
P-4 $(xz)(yz)\leq xy$,
P-5 $x(x(xy))=xy$,
P-6 $xy\leq x$.

A *BCK*-algebra Xsatisfying $(xz)(yz)=(xy)z$, for all $x,y,z\in X$, is said to be *positive implicative*.

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A *BCK*-algebra $X$is *positive implicative* if and only if it satisfies$xy=(xy)y$, for all $x,y\in X$ [8]. A *BCK*-algebra Xsatisfying the identity $x(xy)=y(yx)$, for all $x,y\in X$, is said to be *commutative*. A nonempty subset of a *BCK*-algebra $X$is called an *ideal* of Xif it satisfies

I-1$0\in I$,
I-2$xy\in I, y\in I \Rightarrow x\in I$ for all $x,y\in X$.

A nonempty subset $I$of a *BCK*-algebra $X$is called a *positive implicative ideal* of Xif it satisfies (I-1) and

I-3 $(xy)z\in I, yz\in I \Rightarrow xz\in I$ for all $x,y,z\in X$.

Note that every positive implicative ideal is an ideal (see [12, Proposition 2]). If there is a special element 1in a *BCK*-algebra Xsatisfying $x\leq 1$, for all $x\in X$, then$1$is called a *unit* of X. A *BCK*-algebra X with unit is said to be *bounded*. In what follows let X denote a bounded *BCK*-algebra unless otherwise specified, and we will use the notation x^* instead $1x$ for all $x\in X$. In a bounded *BCK*-algebra Xwe have

P-7 $1^*=0$and $0^*=1$.
P-8 $y\leq x$ implies that $x^*\leq y^*$.
P-9$x^*y^*\leq yx$.

If X is a commutative bounded *BCK*-algebra, then the equalities $(x^*)^*=x$, $x^*y^*=yx$ hold, for all $x,y\in X$, [8].

Definition 2.1. [13] Let F be a nonempty subset of X Then F is called a *filter* of X, if it satisfies the conditions.

F-1 $1\in X$,
F-2 $(x^*)^*y^*\in F, y\in F \Rightarrow x\in F$ for all $x,y\in X$.

Proposition 2.2. [13, Theorem 11] Assume that X is *commutative*. Then a nonempty subset F of X is a *filter* of X if and only if it satisfies F-1 and

F-3 $(xy)^*\in F, x\in F \Rightarrow y\in F$ for all $x,y\in X$.

Definition 2.3. [9] A nonempty subset F of $X$is called a *satisfactory filter* of X, if it satisfies (F-1) and

F-4 $(x(y(yz)^*)^*)^*\in F, x\in F \Rightarrow (yz)^*\in F$ for all $x,y,z\in X$.

We review some fuzzy concepts. A fuzzy subset of a nonempty set X is a function $\mu: X \to [0,1]$.We shall use the notation X_μ for $\{x \in X | \mu(x) = \mu(1)\}$. The set $\mu_t = \{x \in X | \mu(x) \geq t\}$, where $t \in [0,1]$, is called the *t-level subset* of μ .

FUZZY FILTERS

Definition 3.1. [3] A fuzzy subset $\mathcal{F}$in$X$ is said to be a *fuzzy filter* of X, if it satisfies:

FF-1 $\mathcal{F}(1) \geq \mathcal{F}(x)$,
FF-2 $\mathcal{F}(x) \geq min\{\mathcal{F}((x^*y^*)^*), \mathcal{F}(y)\}$, for all $x, y \in X$.

Note that every fuzzy filter is order preserving (see [10, Proposition 3.5]).

Lemma 3.2. [3, Proposition 3.6] Let $\mathcal{F}$ be a fuzzy filter of X.

Then
$\mathcal{F}(x) \geq min\{\mathcal{F}((yx)^*), \mathcal{F}(y)\}$, for all $x, y \in X$ (1)

The following example shows that there is a fuzzy set $\mathcal{F}$ in X such that $\mathcal{F}$ satisfies condition (1), but not a fuzzy filter of X.

Example 3.3. Let $X = \{0, a, b, 1\}$ be a bounded *BCK*-algebra with $*$ defined by

*	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Define $\mathcal{F}$ of $X$by$\mathcal{F}(1) = \mathcal{F}(a) = 0.7$and $\mathcal{F}(0) = \mathcal{F}(b) = 0.5$.Routine calculations give that $\mathcal{F}$ is not a fuzzy filter of$X$.
To consider the converse of Lemma 3.2, we need more strong conditions.

Theorem 3.4. Assume that$X$ is commutative. If $\mathcal{F}$is an order preserving fuzzy set in X satisfying (1), then $\mathcal{F}$ is a fuzzy filter of X .

Proof. Note that$y \leq 1 = 1(0) = (y1)^*$ for all $y \in X$. since $\mathcal{F}$ is order preserving, $\mathcal{F}(y) \leq \mathcal{F}(y1)^*$. using (1), we have

$$\mathcal{F}(1) \geq min\{\mathcal{F}((y1)^*), \mathcal{F}(y)\} \geq min\{\mathcal{F}(y), \mathcal{F}(y)\} = \mathcal{F}(y)$$

for all $y \in X$. Since $x^*y^* = yx$ for all $x, y \in X$, it follows from (1) that

$$\mathcal{F}(x) \geq min\{\mathcal{F}((yx)^*), \mathcal{F}(y)\} = min\{\mathcal{F}(x^*y^*)^*, \mathcal{F}(y)\}$$

Hence $\mathcal{F}$ is a fuzzy filter of X .

Theorem 3.5. Assume that X is commutative. If $\mathcal{F}$ is a fuzzy set in X satisfying FF-1 and (1), then $\mathcal{F}$ is a fuzzy filter of X .
Proof. It is by the proof of Theorem 3.4.

Theorem 3.6. Every fuzzy filter $\mathcal{F}$of Xsatisfies for all $x, y, z \in X$:

$$y \leq (xz)^* \Rightarrow \mathcal{F}(z) \geq min\{\mathcal{F}((x), \mathcal{F}(y)\}$$

Proof. Let $x, y, z \in X$ satisfy $y \leq (xz)^*$ for all $y \in X$. Since $\mathcal{F}$ is order preserving, $\mathcal{F}(y) \leq \mathcal{F}(xz)^*$. it follows from Lemma 3.2 that

$$\mathcal{F}(z) \geq min\{\mathcal{F}(xz)^*, \mathcal{F}(x)\} \geq min\{\mathcal{F}(y), \mathcal{F}(x)\}$$

This completes the proof.

FUZZY SATISFACTORY FILTERS

Definition 4.1. A fuzzy subset $\mathcal{F}$ in X is said to be a *fuzzy filter* of X, if it satisfies:

FF-1 $\mathcal{F}(1) \geq \mathcal{F}(x)$,
FF-2 $\mathcal{F}(x) \geq min\{\mathcal{F}((x^*y^*)^*), \mathcal{F}(y)\}$, for all $x, y \in X$.

Example 4.2. Let $X = \{0, a, b, 1\}$ be a bounded *BCK*-algebra with $*$ defined by

*	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

Define $\mathcal{F}$ of $X$by$\mathcal{F}(1) = 0.6$ and $\mathcal{F}(0) = \mathcal{F}(a) = \mathcal{F}(b) = 0.5$. Routine calculations give that $\mathcal{F}$ is a fuzzy satisfactory filter of X.

Example 4.3. Let $X = \{0, a, b, c, 1\}$ be a bounded *BCK*-algebra with $*$ defined by

*	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	a	0
c	c	c	c	0	0
1	1	c	c	a	0

Define $\mathcal{F}$ of $X$by$\mathcal{F}(1) = \mathcal{F}(c) = 0.6$ and $\mathcal{F}(0) = \mathcal{F}(a) = \mathcal{F}(b) = 0.5$. Routine calculations give that $\mathcal{F}$ is a fuzzy satisfactory filter of X.

We can generalize above examples as follows.

Theorem 4.4. Let F be anon empty subset of X and let $\mathcal{F}_F$ be a fuzzy subset of X , defined by $\mathcal{F}_F(x) = \begin{cases} s & \text{if } x \in F, \\ t & \text{otherwise,} \end{cases}$ for all $x \in X$and $s, t \in [0,1]$ with $s > t$. Then $\mathcal{F}_F$ is a fuzzy satisfactory filter of X if and only if F is a satisfactory filter of X.
Proof. Assume that $\mathcal{F}_F$ is a fuzzy satisfactory filter of X. Obviously $\mathcal{F}_F(1) = s$, and so $1 \in F$. Let $x, y, z \in X$ satisfy $(x(y(yz)^*)^*)^* \in F$ and $1 \in F$. Then $\mathcal{F}_F((x(y(yz)^*)^*)^*) = s$ and $\mathcal{F}_F(x) = s$. it follows form FF-2 that

$$\mathcal{F}_F((yz)^*) \geq \min\{\mathcal{F}_F((x(y(yz)^*)^*)^*), \mathcal{F}_F(x)\} = s$$

So that $(yz)^* \in F$. Hence$F$is a satisfactory filter of X. Conversely suppose that F is a satisfactory filter of X . since $1 \in F$, we have $\mathcal{F}_F(1) = s \geq \mathcal{F}_F(x)$ for all $x \in X$.

Let $x, y, z \in X. If$ $(x(y(yz)^*)^*)^* \in F$and $x \in F$, then $(yz)^* \in F$. Thus

$\mathcal{F}_F((yz)^*) = s = \min\{\mathcal{F}_F((x(y(yz)^*)^*)^*), \mathcal{F}_F(x)\}$

If$(x(y(yz)^*)^*)^* \notin F$ or$x \notin F$, then

$\min\{\mathcal{F}_F((x(y(yz)^*)^*)^*), \mathcal{F}_F(x)\} = t \leq \mathcal{F}_F((yz)^*)$

Therefore $\mathcal{F}_F$is a fuzzy satisfactory filter of X.

Theorem 4.5. Let $\mathcal{F}$ be a fuzzy set of X. Then $\mathcal{F}$ is a fuzzy satisfactory filter of X if and only if $\mathcal{F}_t$ is a satisfactory filter of X where $\mathcal{F}_t \neq \emptyset$.

Proof. Assume that$\mathcal{F}$ is a fuzzy satisfactory filter of X and $\mathcal{F}_t \neq \emptyset$ for all$t \in [0,1]$, Obviously $1 \in \mathcal{F}_t$. Let $x, y, z \in X$satisfy $((x(y(yz)^*)^*)^* \in \mathcal{F}_t$and x $\in \mathcal{F}_t$. using FF-2, we get

$\mathcal{F}((yz)^*) \geq \min\{\mathcal{F}((x(y(yz)^*)^*)^*), \mathcal{F}(x)\} \geq t$

And so $(yz)^* \in \mathcal{F}_t$. Hence $\mathcal{F}_t$ is a satisfactory filter of X. Conversely suppose that $\mathcal{F}_t \neq \emptyset$, for all $t \in [0,1]$. We claim that FF-1 and FF-2 are valid. If FF-1 is not valid. If FF-1 is not valid, $\mathcal{F}(1) < F(x_0)$then for some $x_0 \in X$. Let $t_0 = \frac{1}{2}\{\mathcal{F}(1) + (x_0)\}$. Then $t_0 \in [0,1]$ and $\mathcal{F}(1) < t_0 < F(x_0)$, which implies that$x_0 \in \mathcal{F}_{t_0}$ and so $\mathcal{F}_{t_0} \neq \emptyset$.Since $\mathcal{F}_{t_0}$ is a satisfactory filter of X, $1 \in \mathcal{F}_{t_0}$. Thus $\mathcal{F}(1) \geq t_0$ which is a contradiction. Therefore FF-1 holds. Suppose that FF-2 is false. Then there are$x_0, y_0, z_0 \in X$ such that

$\mathcal{F}((y_0z_0)^*) < \min\{\mathcal{F}((x_0(y_0(y_0z_0)^*)^*)^*), \mathcal{F}(x_0)\} \geq t$

Taking$s_0 = \frac{1}{2}\{\mathcal{F}((y_0z_0)^*) + \min\{\mathcal{F}((x_0(y_0(y_0z_0)^*)^*)^*), \mathcal{F}(x_0)\}\}$, weget$s_0 \in [0,1]$and

$\mathcal{F}((y_0z_0)^*) < s_0 < \min\{\mathcal{F}((x_0(y_0(y_0z_0)^*)^*)^*), \mathcal{F}(x_0)\}$

It follows from the right-hand side of the above inequality that

$(x_0(y_0(y_0z_0)^*)^*)^* \in \mathcal{F}_{s_0}$ *and* $x_0 \in \mathcal{F}_{s_0}$

So from F-4 that $(y_0z_0)^* \in \mathcal{F}_{s_0}$. Thus$\mathcal{F}((y_0z_0)^*) \geq s_0$ so, which is impossible. Hence FF-2 is also valid. Consequently, $\mathcal{F}$ is a fuzzy satisfactory filter of X.

Theorem 4.6. Assume that $X$is commutative. Then every fuzzy filter$\mathcal{F}$of $X$is a fuzzy satisfactory filter of X if and only if satisfies the following inequality for all $x, y \in X$, $\mathcal{F}((xy)^*) \geq \mathcal{F}(x(xy)^*)^*$.

Proof. Assume that $\mathcal{F}$ is a fuzzy satisfactory filter of X. Then the inequality is satisfied straightforward. Conversely suppose that the inequality is satisfied, then by Lemma 3.2., we have
$\mathcal{F}((yz)^*) \geq \mathcal{F}(y(yz)^*)^* \geq \min\{\mathcal{F}((x(y(yz)^*)^*)^*, \mathcal{F}(x)\}$
for all $x, y, z \in X$. Hence $\mathcal{F}$ is a fuzzy satisfactory filter of X.

Theorem 4.7. If $X$is commutative. Then every fuzzy satisfactory filter of X is a fuzzy filter of X .

Proof. Let $\mathcal{F}$ be a fuzzy satisfactory filter of X and let $x, y \in X$. Since $((x)^*)^* = x$, we get $(xy)^* = (x(((y)^*)^*)^*)^*$. It follows from FF-2 that

$\mathcal{F}(y) = \mathcal{F}(((y)^*)^*) \geq \min\{\mathcal{F}((xy)^* = (x(((y)^*)^*)^*)^*)^*, \mathcal{F}(x)\}$

$= \min\{\mathcal{F}((xy)^*, \mathcal{F}(x)\}$

So, from Theorem 3.5. that $\mathcal{F}$ is a fuzzy filter of X .

The converse of Theorem 4.7. may not be true as seen in the following example.

Example 4.8. Let $X = \{0, a, b, 1\}$ be a set with $*$ defined by

*	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	b	a	0

Then X is a bounded *BCK*-algebra (see [14]). Define $\mathcal{F}$ of $X$by$\mathcal{F}(1) = 0.6$ and$\mathcal{F}(0) = \mathcal{F}(a) = \mathcal{F}(b) = 0.5$. Routine calculations give that $\mathcal{F}$ is a fuzzy filter of X. But it is not a fuzzy satisfactory filter of X. In fact, note that $(b(((ba)^*)^*)^*)^* = (b(a)^*)^* = (bb)^* = 0^* = 1$ and $(ba)^* = a^* = b$. Hence

$\mathcal{F}((ba)^*) = 0.6 > 0.5 = \min\{\mathcal{F}((((b(ba)^*)^*)^*)^*, \mathcal{F}(1)\}$

We give conditions for a fuzzy filter of X. is a fuzzy satisfactory filter of X.

Theorem 4.9. Let X be commutative and positive implicative, then every fuzzy filter of X is a fuzzy satisfactory filter of X.

Proof. Let $\mathcal{F}$ be a fuzzy filter of X. Since

$(yz)^* = (z^*y^*)^* = ((z^*y^*)^*y^*)^* = (yz)^*y^*)^* = ((y(yz)^*)^*$

for all $x, y, z \in X$, it follows from Lemma 3.2 that

$\mathcal{F}((yz)^*) = \mathcal{F}(((y(yz)^*)^*) \geq \min\{\mathcal{F}((x(y(yz)^*)^*)^*), \mathcal{F}(x)\}$
so that $\mathcal{F}$ is a fuzzy satisfactory filter of X.

Theorem 4.10. Let X be commutative and$F$ be a filter of X. Then $\mathcal{F}$is a fuzzy satisfactory filter of Xif and only if it satisfies for all $x, y \in X$

$\mathcal{F}((xy)^*) \geq \mathcal{F}(((x(xy)^*)^*)$ (2)

Proof. Let F be a fuzzy satisfactory filter of$X$. Then

$\mathcal{F}((xy)^*) \geq \min\{\mathcal{F}((((x(xy)^*)^*)^*)^*), \mathcal{F}(1)\}$

$= \min\{\mathcal{F}((x(xy)^*)^*), \mathcal{F}(1)\}$

$= \min\{\mathcal{F}((x(xy)^*)^*)$

for all $x, y \in X$. Conversely suppose that $\mathcal{F}$ is a fuzzy filter of X that satisfies (2). Using (2) and Lemma 3.2. we have

$\mathcal{F}((yz)^*) \geq \mathcal{F}(((y(yz)^*)^*) \geq \min\{\mathcal{F}((x(y(yz)^*)^*)^*), \mathcal{F}(x)\}$

Hence $\mathcal{F}$ is a fuzzy satisfactory filter of X.

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