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**Research Article**

# COMPARISON BETWEEN GAUSSIAN AND MEAN CURVATURES ON SMOOTH LOGICALLY CARTESIAN SURFACE MESHES USING MATLAB

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**Abstract**

Estimating intrinsic geometric properties of a surface from a polygonal mesh obtained from range data is an important stage of numerous algorithms in computer and robotvison, computer graphics, geometric modeling, industrial and biomedical engineering. This work considers different computational schemes for local estimation of intrinsic curvature geometric properties. Deferent algorithms and their modifications was tested on geometric models .The aims of this paper is to compare between Gaussian curvature and mean curvature using Matlab. We followed the applied mathematical method using Matlab. We compared the analytically computed values of the Gaussian and mean curvatures in first we found in figure the exact Gaussian curvature imply the verities of sample data like circle shape . In figure 2 the mean curvature imply the vertices of sample data like triangular shape for same data. Second ; in figure 3 we shown that how mean curvature (bubble sample) defined equation one of these bubble is upper and other one is lower .We explained in figure 4 how Gaussian curvature (bubble sample) the same equations.

**Keywords:** Comparison, Gaussian Curvature, Mean Curvature, Smooth Logically Cartesian Surface Meshes, Matlab.

## 1. INTRODUCTION

A special class of objects in differential geometry are surfaces of constant mean curvature. Different notions of a discrete mean curvature may lead to different cmcsurfaces [13]. If $M$ is a compact surface, with planar boundary, embedded in $R^3$with non zero constant mean curvature $H$ and it is contained in one of the half spaces determined by the plane of the boundary, then the Alexandrov's method of reflection shows that $M$ has all the symmetries of its boundary.[10] The concept of *mean curvature* of a surface goes back to Sophie Germain's work on elasticity theory in the seventeenth century. The mathematical formulation of the mean curvature was first derived by Young and then by Laplace in the eighteenth, see The mean curvature of a surface is an extrinsic measure of curvature which locally describes the curvature of surface in some ambient space. [9] Curvature is one of the main important notions used to study the geometry and the topology of a surface. In combinatorial geometry, many attempts to define a discrete equivalent of Gaussian and mean curvatures have been developed for polyhedral surfaces [8]. Basically Delaunay surfaces are the surfaces of revolution with constant mean curvature. [14] A constant mean curvature surface immersed in Euclidean three-space can be viewed as a surface where the exterior pressure and the surface tension forces are balanced. For this reason they are thought of as soap bubbles or films depending on the considered surface being either closed (that is, compact without boundary) or compact with non-empty boundary. [12]

## 2. Plane Curves

Curves are one-dimensional geometric objects which are straight or curved within a higher dimensional ambient space. They are widely used to represent thin physical objects such as rods and wires, as well as to describe the trails of moving objects. Less common but no less interesting examples of curves include singular features under physical processes with concentration mechanisms. e.g. rivers in an eroded terrain or tornadoes in a fluid**.**

Mathematically, one views curves

• Explicitly - a curve is a family of points that can be continuously parametrized by a single variable.

Implicitly - a curve in the plane $R^2$ is the level set $\{(x,y)\in R^2|\, f(x,y)=0\}$ of a continuous scalar function $f: IR^2\rightarrow IR$; a curve in $R^3$ is the intersection of two level set surfaces $f(x,y,z)=0, g(x,y,z)=0$.

These two representations are locally equivalent in generic cases. [1]

## 3. Regular Curves

Experience has shown that it is useful and reasonable to deal in differential geometry (and in other disciplines as well, such as the theory of analytic function $s$ of a complex variable) with a class of plane curves called regular curves. A regular curve is defined as the locus of points) traced out by the end point of a vector $X(t)$ in an $X_1, X_2 - plane$

$$X(t)=(x_1(t),x_2(t)),\qquad \alpha\le t\le\beta$$

and such that $X(t)$ satisfies the following conditions:

(a) $X(t)$ has continuous second derivatives in the interval $\alpha\le t\le\beta$

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(b) $X'$ the derivative of $X(t)$,is now here zero.

These conditions merit some discussion. First of all, it might be noted in connection with the condition (a) that for a good deal of the discussion to follow it would be sufficient to require the existence of a continuous first derivative. In general in this book the existence of a certain finite number of derivatives of the functions employed will be assumed, but the minimum number of derivatives needed from case to case will not always be stated. On the other hand, it is not desirable t o require the functions to be analytic, as is commonly done in the older literature. It is necessary t o operate carefully with the tools of analysis, but it is never the less geometry rather than analysis that is the subject of this research [5].

## 4. Differential Geometry of Curves

Differential geometry is a discipline of mathematics that uses the techniques of calculus and linear algebra to study problems in geometry. The theory of plane, curves and surfaces in the Euclidean space formed the basis for development of differential geometry during the 18th and the 19th century. The core idea of both differential geometry and modern geometrical dynamics lies under the concept of manifold. A manifold is an abstract mathematical space, which locally resembles the spaces described by Euclidean geometry, but which globally may have a more complicated structure [6].

## 5. Curvature

We saw that arc length measures how far a curve is being form stationary. Our preliminary concept of curvature is that it should measure how far a curve is from being a line. For a planar curve the idea used to be to find a circle that best approximates the curve at a point (just like a tangent line is the line that best approximates the curve). The radius of this circle then gives a measure of how the curve bends with larger radius implying less bending. Huygens did quite a lot to clarify this idea for fairly general curves using purely geometric considerations (no calculus) and applied it to the study involutes and evolutes. Newton seems to have been the first to take the reciprocal of this radius to create curvature as we now define it. He also generated some of the formulas in both Cartesian and polar coordinates that are still in use today. [11]

**Definition (5.1)**: Well, a line is not curved at all; its curvature has to be zero. A circle with a small radius is more "curved" than a circle with a large radius. Circles and lines have constant curvature. Curves that are not (pieces of) circles or lines will have a curvature varying from point to point. [6]

**Difinition (5.2)** let $\alpha: I \rightarrow IR^3$ be a curve parametrized by are length $s \epsilon I$ the number $|\alpha''(s)| = k(s)$ *is called curvature* [7]

**Definitions (5.3**) Let $C$ be a regular smooth curve in plane or in space with arc length parametrization $r_{al}: [0, l] \rightarrow IR^i t(s) = r'_{al}$ is a unit tangent vector to the curve at the point $p_s$. The vector function $t:: [0, l] \rightarrow IR^i$ is called the unit tangent vector field moving along the curve. We are now going to analyse the information hidden in the derived vector field $t' = r''_{al}$ along the curve $C$. An application yields:

**Corollary (5.4):** (1) At every point $p_s$ of the curve, the derivative $t'(s)$ is perpendicular to $t(s) = t'(s).t(s) = 0$ .

(2) For a plane curve $C$ the vectors $t'(s)$ *and* $\hat{t}(s)$ are parallel. [4]

## 6. Mean and Gaussian Curvatures

Recall that the trace and determinant of a linear operator are the sum and product of its eigenvalues respectively. These 2 properties of the linear map $dN_p$ the differential of the Gauss map at a point on a regular surface, are used in 2 important definitions of curvature.

**Definition (6.1)** Let$S$ be a regular surface, and $p$ be a point on this surface. Let $k_1$ and $k_2$ denote the principal curvatures (and negative eigenvalues of $dN_p$ ). The Gaussian curvature K of S at p is the determinant of $dN_p$. The mean curvature H of S at P is negative one-half of the trace of $dN_p$. These can be summarized as follows:

$$K = k_1 k_2$$

$$H = \frac{k_1 + k_2}{2}$$

Mean curvature is aptly named since it represents the "average normal curvature" over all directions. Recall from the proof of Euler's theorem that every unit vector $v \in R^2$ can be written as $v = e_1 \cos\theta + e_2 \sin\theta$ where the $e_i$ are the principal directions at a point $p$ on regular surface $S$. We can thus express the normal curvature at $p$ as a function of the angle which $v$ makes with the basis $e_i$ using Euler's Theorem: $k_n(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta$. We then compute the average value of this function for $\theta \in [0, 2\pi)$:

$$\frac{1}{2\pi}\int_0^{2\pi} k_n(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} k_1 \cos^2\theta + k_2 \sin^2\theta \; d\theta$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (k_1 + k_2) + (k_1 - k_2)\cos 2\theta \; d\theta$$

$$H = \frac{k_1 + k_2}{2}$$

The interpretation of Gaussian curvature is less obvious but can be understood by examining surfaces whose Gaussian curvatures have different signs.

**Definition (6.2).** Let $S$ be a regular surface and $p \in S$. Let $K$ denote the Gaussian curvature of $S$ $at$ $p$. This point is called:

- Elliptic if $K > 0$
- Hyperbolic if $K < 0$
- Parabolic if $K = 0, dN_p \neq 0$
- Planar if $K = 0$, $dN_p = 0$.

We can informally think of an elliptic point as curving "the same way" in all directions (the way a sphere does) so that both principal curvatures are of the same sign. A hyperbolic point is one in which the principal curvatures are of opposite signs, so that it resembles a saddle point. A parabolic point has a positive principle curvature in some direction (since the surface is non-planar) but achieves a minimum curvature of 0 in some direction. In a neighborhood of a parabolic point, a surface looks like a curved piece of paper. In a neighborhood of a planar point, a surface just looks flat. [2]

## 7. Constant Mean Curvature Hypersurfaces

Let $\sum$be an orientable $C^2$hypersurface of $R^N$and denote by $\mathcal{V}_{\sum} : \sum \to IR^N$the unit normal vector field on $\sum$ For every $P \in \sum$, we let$\{e_1; : : : ; e_N\}$ be an orthonormal basis of the tangent plane $T_P \sum of \sum at P$. The (normalized) *mean curvature* at p of $\sum$ is givenby

$$H\left(\sum;P\right):=\frac{1}{N-1}\sum_{i=1}^{N-1}(D\,\mathcal{V}_{\sum}(P)e_i,e_i).$$

Here and in the following, $\langle,\rangle$and " ," denote scalar product on $IR^N$. As a consequence, for a $C^1$ -extension of $\mathcal{V}_{\sum}$by a unit vector field $\widetilde{\mathcal{V}_{\sum}}$e in a neighborhood of $p$in $IR^N$, we have

$$H\left(\sum;P\right):=\frac{1}{N-1}div_R^N\widetilde{\mathcal{V}_{\sum}}(P).$$

Let Ω and E be two open subsets of $R^N$with E ⊂ Ω. Then the *perimeter functional of* E*relative to* Ω (total variation of 1E in Ω) is given by

$$P(E,\Omega)=|DI_E|(\Omega):=\sup\{\int_E div\xi(x)\,dx:\xi \in C_c^\infty(\Omega;R^N)|\xi|\leq 1\}$$

In the following, we simply write $P(E) := P(E, IR^N)$.

**Definition (7.1)** We consider a vector field $\xi \in C_c^\infty(IR^N, IR^N)$and define the flow $(Y_t)_t \in IR(Yt)$

$$\begin{cases}\partial_t Y_t(x)=\xi(Y_t(x)) & t\epsilon\, IR\\ Y_0(x)=x & \text{for all } x\epsilon\, IR^N\end{cases}$$

For $E \subset IR^N$, we call the family of sets $E_t := Y_t(E), t\,\epsilon\, IR$, the variation of E with respect to the vector field ξ[3]

**Proposition (7.2):** Let Ω and E be two bounded domains of $R^N$, with E of class $C^2$. Let$\lambda\,\epsilon$ and$(E_t)_t$be a variation of E with respect to $\xi \in C_c^\infty(\Omega;R^N)$. Then the map

$$t \mapsto J(t) := P(E_t,\Omega)-\lambda|E_t \cap \Omega|$$

is differentiable at zero. Moreover

$$J'(0)=(N-1)\int_{\partial E}\{H(\partial E;P)-\lambda\}v(P)dV(P),$$

Where $v(P) := \langle\xi(P), \upsilon_{\partial E}(P)\rangle$ and $\upsilon_{\partial E}$ is the unit exterior normal vector field of E.

**Theorem (7.3):** An embedded closed C2 hyper surface in RN , with nonzero constant mean curvature is a finite union of disjoint round spheres with same radius.

**Lemma (7.4):** Let $\omega\,\epsilon\, C^2(B)\cap C^1(B)$ be a nonnegative function on B and satisfy

$$\partial_i\left(a_{ij}(x)\,\partial_j\, w\right)=0 \text{ on B}$$

for some positive definite matrix $a_{ij}$ of class $C^1$. Then the following holds.

i. If $w(x_0) = 0$, for some $x_0 \,\epsilon\, \partial B$ then $w = 0$ in B .
ii. If $w(x_0) = \nabla w(x_0).v = 0$ for some $x_0\,\epsilon\,\partial B$ and v a unit vector normal to $T_{x0}\ \partial B$, then w = 0 in B.

**Theorem (7.5):** *There exist* $b_0, h_* > 0$ *and a smooth curve* $(-b_0, b_0) \ni b \mapsto \lambda(b)$*such*

*That* $\lambda(0) = 1$ *and* $\varphi_b(t)=\frac{1}{h_*}+\frac{b}{\lambda(b)}\{\cos(\lambda(b)t)+v_a(\lambda(b)t)\}$,

*Where*

$$v_b \to 0 \text{ in } C^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z}) \text{ as } b\to 0 \text{ and } \int_{-\pi}^{\pi}v_b(t)\cos(t)\,dt=0$$

*for every* $b\,\epsilon\,(-b_0, b_0)$ .

**Remark (7.6):** *We note that the family of surfaces* $(\sum_b)_{b>1}$*are the immersed constant meancurvature surfaces known as the nodoids.*[9]

## 8. Gaussian (GC) and Mean Curvatures (MC) Code

**Function curvatures file code:**

```
function [GC MC]= curvatures(x,y,z,tri)
tri3d=triangulation(tri,[x,y,z]);
bndry_edge=freeBoundary(tri3d);
f_normal = faceNormal(tri3d);
f_center = incenter(tri3d);
for i=1:length(tri(:,1))
v1=[x(tri(i,2))-x(tri(i,1)),y(tri(i,2))-y(tri(i,1)),z(tri(i,2))-z(tri(i,1))];
v2=[x(tri(i,3))-x(tri(i,1)),y(tri(i,3))-y(tri(i,1)),z(tri(i,3))-z(tri(i,1))];
area_tri(i,1)=0.5*norm(cross(v1,v2));
end
for i=1:length(tri(:,1))
p1=tri(i,1);
p2=tri(i,2);
p3=tri(i,3);
v1(i,:)=[x(p2)-x(p1),y(p2)-y(p1),z(p2)-z(p1)];
v2(i,:)=[x(p3)-x(p2),y(p3)-y(p2),z(p3)-z(p2)];
v3(i,:)=[x(p1)-x(p3),y(p1)-y(p3),z(p1)-z(p3)];
l_edg(i,1)=norm(v1(i,:));
l_edg(i,2)=norm(v2(i,:));
l_edg(i,3)=norm(v3(i,:));
ang_tri(i,1)=acos(dot(v1(i,:)/l_edg(i,1),-v3(i,:)/l_edg(i,3)));
ang_tri(i,2)=acos(dot(-v1(i,:)/l_edg(i,1),v2(i,:)/l_edg(i,2)));
ang_tri(i,3)=pi-(ang_tri(i,1)+ang_tri(i,2));
end
a_mixed=zeros(1,length(x));
alf=zeros(1,length(x));
GC=zeros(length(x),1);
MC=zeros(length(x),1);
for i=1:length(x)
mc_vec=[0,0,0];
n_vec=[0,0,0];
if find(bndry_edge(:,1)==i)
else
```

```
clearneib_tri
neib_tri=vertexAttachments(tri3d,i);
for j=1:length(neib_tri{1})
neib=neib_tri{1}(j);
for k=1:3
if tri(neib,k)==i
alf(i)=alf(i)+ ang_tri(neib,k);
break;
end
end
if   k==1
mc_vec=mc_vec+(v1(neib,:)/tan(ang_tri(neib,3))-
v3(neib,:)/tan(ang_tri(neib,2)));
elseif k==2
mc_vec=mc_vec+(v2(neib,:)/tan(ang_tri(neib,1))-
v1(neib,:)/tan(ang_tri(neib,3)));
elseif k==3
mc_vec=mc_vec+(v3(neib,:)/tan(ang_tri(neib,2))-
v2(neib,:)/tan(ang_tri(neib,1)));
end
if(ang_tri(neib,k)>=pi/2)
a_mixed(i)=a_mixed(i)+area_tri(neib)/2;
else
if (any(ang_tri(neib,:)>=pi/2))
a_mixed(i)=a_mixed(i)+area_tri(neib)/4;
else
sum=0;
for m=1:3
if m~=k
ll=m+1;
ifll==4
ll=1;
end
sum=sum+(l_edg(neib,ll)^2/tan(ang_tri(neib,m)));
end
end
a_mixed(i)=a_mixed(i)+sum/8;
end
end
wi=1/norm([f_center(neib,1)-x(i),f_center(neib,2)-
y(i),f_center(neib,3)-z(i)]);
n_vec=n_vec+wi*f_normal(neib,:);
end
GC(i)=(2*pi()-alf(i))/a_mixed(i);
mc_vec=0.25*mc_vec/a_mixed(i);
n_vec=n_vec/norm(n_vec);
if dot(mc_vec,n_vec) <0
MC(i)=-norm(mc_vec);
else
MC(i)=norm(mc_vec);
end
end
end
```

**Example Test Code:**

```
clc
clearall
a=1;
b=1;
c=1;
n=0;
[X,Y] = meshgrid(0:0.1:0.5, 0:0.1:0.5);
Z=c*((Y./b)^2-(X./a)^2);
for x1=-0.5:0.01:0.5
for y1=-0.5:0.01:0.5
z1=c*((y1/b)^2-(x1/a)^2);
n=n+1;
x(n,1)=x1;
y(n,1)=y1;
z(n,1)=z1;
GC_ex(n,1)=4*a^6*b^6/(c^2*((a^4*b^4/c^2)+4*b^4*x1^2+4*
a^4*y1^2)^2);
MC_ex(n,1)=-(-a^2+b^2-
4*x1^2/a^2+4*y1^2/b^2)/(a^2*b^2*(1+4*x1^2/a^4+4*y1^2/b
^4)^1.5);
end
end
tri=delaunay(x,y);
[GC,MC]=curvatures(x,y,z,tri);
img=figure(1);
clf
set(img, 'Position', [100 100 600 600]);
holdon
axisequal
pp=
patch('Faces',tri,'Vertices',[x,y,z],'FaceVertexCData',GC,'Face
Color','interp','EdgeColor','none');
caxis([-4 , -0.4])
colormapjet
colorbar
xlabel('x')
ylabel('y')
title('Estimated GC');
img=figure(2);
clf
set(img, 'Position', [300 100 600 600]);
holdon
axisequal
pp=
patch('Faces',tri,'Vertices',[x,y,z],'FaceVertexCData',MC,'Face
Color','interp','EdgeColor','none');
caxis([-0.36 , 0.36])
colormapjet
colorbar
xlabel('x')
ylabel('y')
title('Estimated MC');
img=figure(3);
clf
set(img, 'Position', [500 100 600 600]);
holdon
axisequal
pp=
patch('Faces',tri,'Vertices',[x,y,z],'FaceVertexCData',GC_ex,'Fa
ceColor','interp','EdgeColor','none');
caxis([-4 , -0.4])
colormapjet
colorbar
xlabel('x')
ylabel('y')
title('Exact GC');
img=figure(4);
clf
set(img, 'Position', [700 100 600 600]);
holdon
axisequal
pp=
patch('Faces',tri,'Vertices',[x,y,z],'FaceVertexCData',MC_ex,'F
aceColor','interp','EdgeColor','none');
```

```
caxis([-0.36 , 0.36])
colormapjet
colorbar
xlabel('x')
ylabel('y')
title('Exact MC');
```

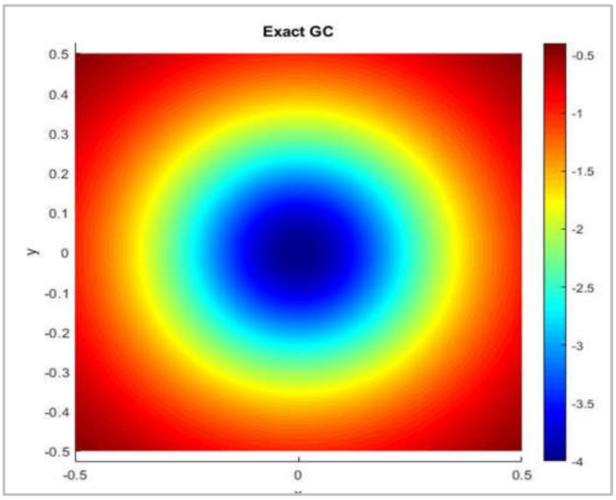


**Fig. 1. Showing how exact Gaussian curvature like circle shape**

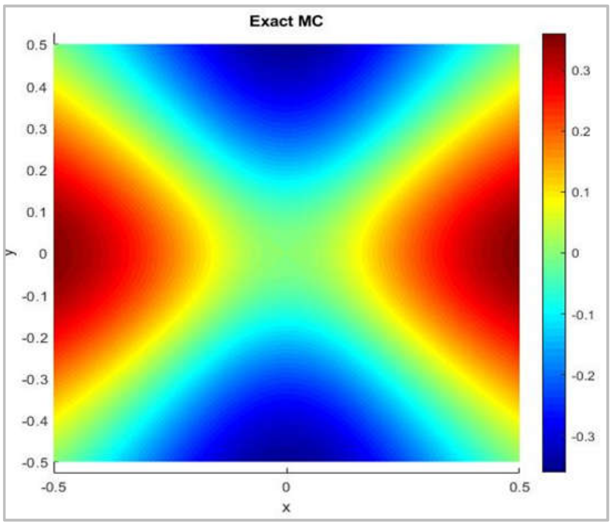


**Fig. 2. Showing mean curvature triangular shape**

**Gaussian (GC) and mean curvatures (MC) Code 2:**

```
function [K,H] = curvature(S)

[fx,fy] = gradient(S);
[fxx,fxy] = gradient(fx);
[~,fyy] = gradient(fy);
K = (fxx.*fyy - fxy.^2)./((1 + fx.^2 + fy.^2).^2);
H = ((1+fx.^2).*fyy + (1+fy.^2).*fxx - 2.*fx.*fy.*fxy)./...
((1 + fx.^2 + fy.^2).^(3/2));
```

- **Example Test2:**

```
f = @(mu1,mu2,s1,s2,x,y) exp(-(x-mu1).^2/(s1.^2)-(y-mu2).^2/(s2.^2));
[X,Y] = meshgrid(linspace(-5,5,200));
S = f(-2,0,2,2,X,Y) - f(2,0,2,2,X,Y)
figure; mesh(S);
[K,H] = curvature(S);
figure;mesh(K); title('Gaussian Curvature','FontSize',20);
figure;mesh(H); title('Mean Curvature','FontSize',20);
```

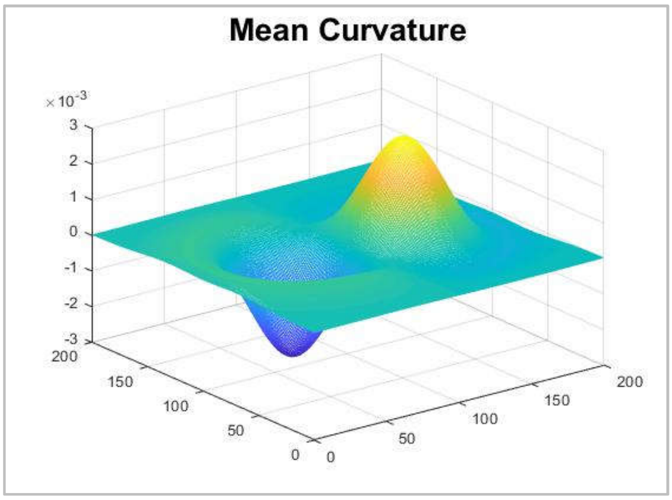


**Fig. 3. Mean curvature (bubble sample)**

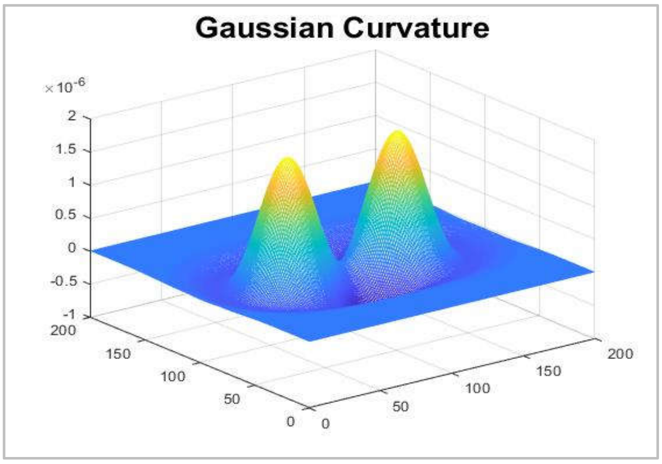


**Fig. 4. Show how Gaussian curvature (bubble sample)**

## RESULTS

Firstly: figure 1 shown how exact Gaussian curvature imply the vertices of sample data like circular or circle shape and figure 2 shown how mean curvature imply the vertices of sample data like triangular shape for same data.

Secondly: figure 3 shown how mean curvature (bubble sample) for defined equation $e^{\left(-\frac{(x-mu\,1)^2}{s_1{}^2}\right)-\frac{(y-mu\,2)^2}{s_1^2}}$ one of these bubble is upper and other one is lower , figure 4 shown how Gaussian curvature (bubble sample) for the same equation the two bubble is upper and the normal curvature in figure 5 is vice versa of mean curvature.

### Conclusion

In this paper we have provided four different algorithms for curvature estimation comparison between the Gaussian and mean curvatures and we found in the two figures(1,2) the figure 1showing how exact Gaussian curvature imply the vertias of sample data like circle shape In figure 2 showing how mean curvature imply the vertices of sample data like triangular shape for same data n figure 3 shown that how mean curvature for defined equation one of these bubble is upper and other one is lower .In figure 4 explained how Gaussian curvature for the same equations. The experimental results we presented validate the approach and show that curvature behaves better than some mostly used techniques.

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