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Research Article

COMPARATIVE STUDY OF THE THREE FREQUENT ENCOUNTERED NUMERICAL METHODS OF INTEGRATION WITH EMPHASIS ON THEIR ERROR TERMS

[1, *]Njipouakouyou Samuel and [2]Mahamat Saleh Daoussa Haggar

[1]Frmr senior lecturer, Faculty of Sciences, University of Dschang, Cameroon
[2]Maître de Conférences, Dept de Mathématiques, Fac des Sciences Exactes et Appliquées, Laboratoire de Modélisation, Mathématiques, Informatique, Applications et Simulation (L2MIAS), Université de Ndjamena – TCHAD

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Abstract

Very few functions can be integrated analytically. For the large class of those ones whose integral cannot be found such a way, only numerical methods are helpful. They are many and all of them generate errors. The goal of this paper is to find out those ones which minimize these errors. The most encountered numerical methods the Newton-Cotes, Tchebyshev and Gauss. This study indicates that the method of Gauss gives better results, follows by the method of Tchebyshev. The method of Newton-Cotes is at the third position. Moreover, for more accurate results the integrating function should be replace by an algebraic function of order not exceeding 2.

Keywords: Integration, numerical methods, finite integral, domain of integration, integrating function, integral, linear combination, sub domains of integration, nods of integration, error terms, estimations of the errors.

1. Position of the problem

The fast and continuous development of sciences has enabled researchers to elaborate more complicated models to describe as closer as possible natural phenomena. They usually lead to computations of finite integrals of the form:

$$I = \int_A^B f(x)dx, \quad A \le x \le B. \qquad \textbf{(1)}$$

Very few integrals of (1) are analytically treated. Many of them can only be computed through numerical methods, (Melentev, 1962, Patel, 1994). Interval (A,B) is the domain of integration, f(x) – the integrating function, I– the integral. The goal of this study is to develop and compare between themselves some methods which should enable us to easier compute integral (1) when it cannot be done analytically.

For this purpose, I is expressed as a linear combination of the integrating function calculated at some special nods of integration, x_k, with $A \le x_k \le B$, $1 \le k \le n$:

$$I = C_1y_1 + C_2y_2 + \dots + C_ny_n, \qquad \textbf{(2)}$$

C_k are the constants to be determined, $y_k = f(x_k)$, Formula (2) is elaborated for the domain of integration $-1 \le x \le 1$. For other domains, a new variable u to bring back to the previous domain is introduced as follows:

$$u = \frac{B-A}{2}x + \frac{B+A}{2}, \quad -1 \le u \le 1. \qquad \textbf{(3)}$$

To easy the finding of (1), put y = f(x) as an unlimited algebraic expression of the form:

$$y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots \qquad \textbf{(4)}$$

Clear that (4) should be always limited to a finite number of terms, whence the errors in the results. To elaborate different numerical methods of integration, relations (2), (3) and (4) are considered. This article is not to develop a lot of formulas in each numerical method, but to show the readers how to find the nods of integration x_k and the coefficients C_k in (2).

2. Fundamental basis of the numerical methods of integration

In (1), A=-1, B=1, I and f(x) expressed respectively by (2) and (4). Thus, the classical method of integration gives:

$$I = \int_{-1}^{1} f(x)dx = \int_{-1}^{1}(a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n + \cdots)dx = (a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \dots)I_{-1}^{1} = 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + \dots + \frac{2}{2m+1}a_{2m} + \dots, \quad m = 0, 1, 2, \dots. \qquad \textbf{(5)}$$

Corresponding Author:** ***Njipouakouyou Samuel; *Frmr senior lecturer, Faculty of Sciences, University of Dschang, Cameroon*

The numerators of all the coefficients of the last expression of (5) are 2 and their denominators are in arithmetic progression with 1 the first term and 2 the ratio. The general term of this progression is 2m+1. Reorganizing all the terms, we come to the expression:

$$C_1y_1 + C_2y_2 + \dots + C_ny_n = C_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + \dots + a_nx_1^n + \dots\ ) +$$
$$+ C_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3 + \dots + a_nx_2^n + \dots\ ) +$$
$$+ C_3(a_0 + a_1x_3 + a_2x_3^2 + a_3x_3^3 + \dots + a_nx_3^n + \dots\ ) +$$
$$+ \dots\dots\dots\dots\dots\dots +$$
$$+ C_n(a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + \dots + a_nx_n^n + \dots\ ) +$$
$$+ \dots\dots\dots\dots\dots\dots = \qquad \textbf{(6)}$$
$$a_0(C_1 + C_2 + C_3 + \dots + C_n) +$$
$$+ a_1(C_1x_1 + C_2x_2 + C_3x_3 + \dots + C_nx_n) +$$
$$+ a_2(C_1x_1^2 + C_2x_2^2 + C_3x_3^2 + \dots + C_nx_n^2) +$$
$$+ a_3(C_1x_1^3 + C_2x_2^3 + C_3x_3^3 + \dots + C_nx_n^3) +$$
$$+ \dots\dots\dots\dots\dots\dots +$$
$$+ a_n(C_1x_1^n + C_2x_2^n + C_3x_3^n + \dots + C_nx_n^n) +$$
$$+ \dots\dots\dots\dots\dots\dots =$$
$$= 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + \frac{2}{7}a_6 + \dots$$

Equations (6) must be verified for any a_k. For this condition to be filled, the next systems of equations must be verified:
The first system:

$$C_1 + C_2 + C_3 + \dots + C_n = 2,$$
$$C_1x_1^2 + C_2x_2^2 + C_3x_3^2 + \dots + C_nx_n^2 = \frac{2}{3},$$
$$C_1x_1^4 + C_2x_2^4 + C_3x_3^4 + \dots + C_nx_n^4 = \frac{2}{5},$$
$$C_1x_1^6 + C_2x_2^6 + C_3x_3^6 + \dots + C_nx_n^6 = \frac{2}{7}, \qquad \textbf{(7)}$$
$$\dots\dots\dots\dots\dots\dots$$
$$C_1x_1^{2m} + C_2x_2^{2m} + C_3x_3^{2m} + \dots + C_nx_n^{2m} = \frac{2}{2m+1}.$$

The second one:

$$C_1x_1 + C_2x_2 + C_3x_3 + \dots + C_nx_n = 0,$$
$$C_1x_1^3 + C_2x_2^3 + C_3x_3^3 + \dots + C_nx_n^3 = 0,$$
$$C_1x_1^5 + C_2x_2^5 + C_3x_3^5 + \dots + C_nx_n^5 = 0, \qquad \textbf{(8)}$$
$$\dots\dots\dots\dots\dots\dots,$$
$$C_1x_1^{2m+1} + C_2x_2^{2m+1} + C_3x_3^{2m+1} + \dots + C_nx_n^{2m+1} = 0,$$

It is obvious that for (7) and (8) to be verified, the nods x_k and coefficients C_k must fulfill the conditions:

$x_k = -x_{n-k}$, **(9)** $C_k = C_{n-k}$. **(10)**

Because of the limitation of the number of terms in (4), each equation of (7) should contain an error ε. Thus, these equations should be rewritten as follows:

$$C_1 + C_2 + C_3 + \dots + C_n = 2 + \varepsilon_0,$$
$$C_1x_1^2 + C_2x_2^2 + C_3x_3^2 + \dots + C_nx_n^2 = \frac{2}{3} + \varepsilon_2,$$
$$C_1x_1^4 + C_2x_2^4 + C_3x_3^4 + \dots + C_nx_n^4 = \frac{2}{5} + \varepsilon_4,$$
$$C_1x_1^6 + C_2x_2^6 + C_3x_3^6 + \dots + C_nx_n^6 = \frac{2}{7} + \varepsilon_6, \qquad \textbf{(11)}$$
$$\dots\dots\dots\dots\dots\dots$$
$$C_1x_1^{2m} + C_2x_2^{2m} + C_3x_3^{2m} + \dots + C_nx_n^{2m} = \frac{2}{2m+1} + \varepsilon_{2m},$$

In the theory of numerical integration, the error is usually expressed by a formula of the form (Melentev, 1962, Patel, 1994):

$$\varepsilon = kf^{(s)}(\xi), \qquad \textbf{(12)}$$

k being a constant and ξ - a number such that $-1< \xi < 1$. (12) does not always give good estimations of the error generated as the exact position of ξ inside the domain of integration is unknown. To fix our mind, consider that two different numerical methods have led to a same error term of the form:

$$\varepsilon = \frac{1}{1000}f^{(5)}(\xi),\ -1< \xi < 1 \quad \textbf{(13)}$$

where the integrating function is:

$f(x) = \frac{1}{x+2}$. **(14)**

The fifth order derivative of (14) is:

$f^{(5)}(x) = -\frac{5!}{(x+2)^6}$. **(15)**

If ξ is near x = -1, then x+2 will be closer to 1 and we have:

$\varepsilon \approx -\frac{5!}{1000\,.\,1^6} \approx -\frac{1}{8}$. **(16)**

If ξ is closer to x = 1 and proceeding by analogy, we have:

$\varepsilon \approx -\frac{5!}{1000\,.\,3^6} \approx -\frac{1}{6000}$. **(17)**

Comparing (16) and (17) leads to the conclusion that the first is about 10^3 times larger the second. When the integrating function is not given analytically, this error estimation becomes more complicated and probably impossible, particularly when f(x) is given graphically.

Thus, other methods to evaluate ε should be performed. Clearly that ε depends on the number of nods of integration. Obviously, if the errors in the second members of (11) are minimal, the final error will also be minimal. Considering this remark and putting
$S_{2m} = C_1 x_1^{2m} + C_2 x_2^{2m} + C_3 x_3^{2m} + \ldots + C_n x_n^{2m}$, **(18)**

the error generated could be estimated by the formula::

$\varepsilon_{2m} = S_{2m} - \frac{2}{2m+1}$ **(19)**

permitting us to decide how many nods of integration to be considered to minimize (19).
The present study is based on the analysis of (6) – (10) and (18).

3. Elaboration of the five firsts formulas of each numerical method of integration

The domain of integration is [-1, 1]. It is divided into n equal sub domains of length $\frac{2}{n}$ each. The most encountered numerical methods of integration could be divided into the three next groups.

First group: The nods of integration are the extremities of the sub domains. Here belong the formulas of Newton-Cotes. Sometimes, the middles of sub domains are taken as nods of integration.

Second group: The nods of integration are determined such that all the coefficients C_k in (2) are equal between themselves. The formulas of Tchebyshev are here.

Third group: Here both the nods of integration x_k and the coefficients C_k are obtained solving the system of equations (7). To this group belong the formulas of Gauss.

This study is on the three groups. In each one, formulas for only the five firsts nods of integration will be developed, i.e. n = 1, 2, 3, 4 and 5, their corresponding error calculated and compared between themselves for pointing out the best accurate one to be recommended for operational works.

3.1 Formulas of Newton-Cotes

Trapezoidal and the formula of Simpson are special cases of the formulas of Newton-Cotes respectively for n = 1 and 2, meaning that the domain of integration has been divided into 1 and 2 sub domains. In the interval [-1,1], they are respectively given by the formulas:

$I \approx \frac{2}{n}(\frac{1}{2}y_1 + y_2 + y_3 + \ldots + y_n + \frac{1}{2}y_{n+1})$, **(20)**

(trapezoidal formula),

$I \approx \frac{2}{3n}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \ldots + 2y_{n-1} + 4y_n + y_{n+1})$ **(21)**
(formula of Simpson).

In other domain of integration different from above, say [A, B], these expressions become:

$I \approx \frac{B-A}{n}(\frac{1}{2}y_1 + y_2 + y_3 + \ldots + y_n + \frac{1}{2}y_{n+1})$, **(22)**

and

$I \approx \frac{B-A}{3n}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \ldots + 2y_{n-1} + 4y_n + y_{n+1})$ **(23)**

Put I the exact value of (1), I_n and I_{2n} their approximations when n and 2n sub domains with Δ_n and Δ_{2n} their corresponding errors, we may write $I_n = I + \Delta_n$ and $I_{2n} = I + \Delta_{2n}$ whence $I = I_{2n} - \Delta_{2n}$, Experimentally, it was proved that $\Delta_n \approx 4\Delta_{2n}$, Therefore $I = I_{2n} - \Delta_{2n} \approx I_n - 4\Delta_{2n}$ whence:

$\Delta_{2n} \approx \frac{I_n - I_{2n}}{3}$. **(24)**

It comes that

$I \approx I_{2n} - \frac{I_n - I_{2n}}{3}$ **(25)**

which is the modified trapezoidal formula. (25) gives more accurate results than (22).

Formula of Newton-Cotes for n = 3

[-1, 1] is divided into n = 3 equal sub domains. The nods of integration are $x_1 = -1$, $x_2 = -\frac{1}{3}$, $x_3 = \frac{1}{3}$, $x_4 = 1$. Thus (2) gives:

$I = C_1y_1 + C_2y_2 + C_3y_3 + C_4y_4$. **(26)**

From (10), $C_1 = C_4$ and $C_2 = C_3$. Thus, (7) contains only two unknowns leading to the two equations to be solved for C_1 and C_2:

$C_1 + C_2 = 1$ and $C_1 + \frac{1}{9}C_2 = \frac{1}{3}$,

whence $C_1 = C_4 = \frac{1}{4}$, $C_2 = C_3 = \frac{3}{4}$.

Thus, based on (26), we have the formula of Newton-Cotes for n = 3 in [-1, 1]:

$I = \frac{1}{4}(y_1 + 3y_2 + 3y_3 + y_4)$. **(27)**

For any other domain of integration [A, B], (27) becomes:

$I = \frac{B-A}{8}(y_1 + 3y_2 + 3y_3 + y_4)$. **(28)**

Putting $n = 3n_1$ and proceeding the same way we have the general formula of Newton-Cotes:

$I \approx \frac{3(B-A)}{8n}(y_1+3y_2+3y_3+2y_4+3y_5+3y_6+2y_7+ \ldots + 2y_{n-2}+3y_{n-1}+3y_n+ 2y_{n+1})$ **(29)**

Formula of Newton-Cotes for n = 4

The nods of integration are $x_1= -1$, $x_2 = -\frac{1}{2}$, $x_3 = 0$, $x_4 = \frac{1}{2}$, $x_5 = 1$ and the coefficients C_1, C_2, C_3, C_4 and C_5. From (10), we have $C_1 = C_5$, $C_2 = C_4$ and C_3. So (7) leads to the next system of three equations with three unknowns:

$C_1+C_2+C_3+C_4+C_5 = 2$, or $2C_1+2C_2+C_3 = 2$,
$C_1x_1^2+ C_2x_2^2+ C_3x_3^2+ C_4x_4^2+ C_5x_5^2 = \frac{2}{3}$, or $2C_1x_1^2+ 2C_2x_2^2+ C_3x_3^2= \frac{2}{3}$,
$C_1x_1^4+ C_2x_2^4+ C_3x_3^4+ C_4x_4^4+ C_5x_5^4 = \frac{2}{5}$, or $2C_1x_1^4+ 2C_2x_2^4+ C_3x_3^4 = \frac{2}{5}$.

Replacing x_k by their corresponding values leads to the next system of equations:

$2C_1+2C_2+C_3 = 2$; $2C_1+ \frac{2}{4}C_2= \frac{2}{3}$; $2C_1+ \frac{2}{16}C_2 = \frac{2}{5}$, **(30)**
whence $C_1 = C_5 = \frac{7}{45}$, $C_2 = C_4 = \frac{32}{45}$, $C_3 = \frac{12}{45}$,

and the formula of Newton-Cotes for n = 4 and any domain of integration, [A, B]:

$I \approx \frac{B-A}{90}(7y_1 + 32y_2 + 12y_3 + 32y_4 + 7y_5)$. **(31)**

Formula of Newton-Cotes for n = 5

The nods of integration $x_1 = -1$, $x_2 = -\frac{3}{5}$, $x_3 = -\frac{1}{5}$, $x_4 = \frac{1}{5}$, $x_5 = \frac{3}{5}$, $x_6 = 1$ and the coefficients C_1, C_2, C_3, C_4, C_5 and C_6. From (10), $C_1 = C_6$, $C_2 = C_5$ and $C_3 = C_4$. (7) leads to the next system of three equations with three unknowns to be solved for the coefficients:

$C_1+C_2+C_3+C_4+C_5+C_6 = 2$, $2(C_1+C_2+C_3) = 2$,
$C_1x_1^2+ C_2x_2^2+ C_3x_3^2+ C_4x_4^2+ C_5x_5^2+C_6x_6^2 = \frac{2}{3}$, or $2(C_1x_1^2+ C_2x_2^2+ C_3x_3^2)= \frac{2}{3}$,
$C_1x_1^4+ C_2x_2^4+ C_3x_3^4+ C_4x_4^4+ C_5x_5^4 +C_6x_6^4= \frac{2}{5}$, or $2(C_1x_1^4+ C_2x_2^4+ C_3x_3^4)= \frac{2}{5}$,

Replacing x_k by their corresponding values, the system of equations becomes:

$C_1+C_2+C_3 = 1$, $C_1+ \frac{9}{25}C_2+ \frac{1}{25}C_3= \frac{1}{3}$, $C_1+ \frac{81}{625}C_2+ \frac{1}{625}C_3= \frac{1}{5}$, **(32)**
whence $C_1 = C_6 = \frac{19}{144}$, $C_2 = C_5 = \frac{75}{144}$, $C_3 = C_4 = \frac{50}{144}$,

and the formula of Newton-Cotes for n = 5 and any domain of integration, [A, B]:

$I \approx \frac{B-A}{288}(19y_1 + 75y_2 + 50y_3 + 50y_4 + 75y_5 + 19y_6)$. **(33)**

3.2 The formulas of Tchebyshev

Here all the coefficients in (2) are equal. Thus, we may write that

$C_1 + C_2 + C_3 + \ldots + C_n = nC = 2$, **(39)**

where n is the number of the nods of integration. Thus, the coefficient is $C = \frac{2}{n}$. Taking (9) into consideration, (7) becomes:

$x_1^2+x_2^2+x_3^2+ \ldots +x_{\frac{n}{2}}^2 = \frac{n}{6}$,
$x_1^4+x_2^4+x_3^4+ \ldots +x_{\frac{n}{2}}^4 = \frac{n}{10}$, **(40)**
$x_1^6+x_2^6+x_3^4+ \ldots +x_{\frac{n}{2}}^4 = \frac{n}{14}$,
…………………………

Remark that if the number of nods of integration is odd, then the nod at the middle of [-1, 1], will be $x_{\frac{n+1}{2}} = 0$, the center of symmetry.

Putting $x_k^2 = u_k$, (40) becomes:

$u_1 +x_2 +u_3 + \ldots +u_{\frac{n}{2}} = \frac{n}{6}$,
$u_1^2+x_2^2+x_3^2+ \ldots +x_{\frac{n}{2}}^2 = \frac{n}{10}$, **(41)**
$x_1^3+x_2^3+x_3^3+ \ldots +x_{\frac{n}{2}}^3 = \frac{n}{14}$,
…………………………

Putting $\frac{n}{6} = k_1$, $\frac{n}{10} = k_2$, $\frac{n}{14} = k_3$, …, $\frac{n}{4m+2} = k_n$, (41) becomes:

$u_1 +x_2 +u_3 + \ldots +u_{\frac{n}{2}} =k_1$,
$u_1^2+x_2^2+x_3^2+ \ldots +x_{\frac{n}{2}}^2 =k_2$, **(42)**
$x_1^3+x_2^3+x_3^3+ \ldots +x_{\frac{n}{2}}^3 =k_3$,
…………………………

Remark that the denominators of the second members of equations of (41) are in arithmetic progression, the first term being 6 and the ration 4. Its general term is 4m+2. Assume that (42) has s rows for s roots to be determined. This leads to the next equation of order s to be solved for u:

$u^s + A_1u^{s-1} + A_2u^{s-2} + A_3u^{s-3} + \ldots + A_{s-1}u + A_s = 0$. **(43)**

As by hypothesis $u_1, u_2, \ldots, u_s$ are the roots of (43), they should verify the system of equations:

$$u_1^s + A_1 u_1^{s-1} + A_2 u_1^{s-2} + \ldots + A_{s-1}u_1 + A_s = 0,$$
$$u_2^s + A_1 u_2^{s-1} + A_2 u_2^{s-2} + \ldots + A_{s-1}u_2 + A_s = 0, \quad \textbf{(44)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$u_s^s + A_1 u_s^{s-1} + A_2 u_s^{s-2} + \ldots + A_{s-1}u_s + A_s = 0,$$

Summing (44) member by member leads to the next equation:

$$(u_1^s + u_2^s + u_3^s + \ldots + u_s^s) + A_1(u_1^{s-1} + u_2^{s-1} + u_3^{s-1} + \ldots + u_s^{s-1}) +$$
$$A_2(u_1^{s-2} + u_2^{s-2} + u_3^{s-2} + \ldots + u_s^{s-2}) + \ldots + A_{s-1}(u_1 + u_2 + u_3 + \ldots + u_s)$$
$$+ sA_s = 0. \quad \textbf{(45)}$$

Taking the second members of (42) into account, (45) becomes:

$$k_s + A_1k_{s-1} + A_2k_{s-2} + A_3k_{s-3} + \ldots + A_{s-1}k_1 + sA_s = 0, \quad \textbf{(46)}$$

whence the coefficients of (46) to be obtained from the next formula:

$$A_s = -\frac{1}{s}(k_s + A_1k_{s-1} + A_2k_{s-2} + A_3k_{s-3} + \ldots + A_{s-1}k_1). \quad \textbf{(47)}$$

Following (47), we have the searched coefficients:

$$A_1 = -k_1, \ (s = 1),$$
$$A_2 = -\frac{1}{2}(k_2 + A_1k_1), \ (s = 2),$$
$$A_3 = -\frac{1}{3}(k_3 + A_1k_2 + A_2k_1), \quad (s = 3), \quad \textbf{(48)}$$
$$A_4 = -\frac{1}{3}(k_4 + A_1k_3 + A_2k_2 + A_3k_1), \ (s = 4),$$

and so on. Based on (48) and (46) we could determine the nods of integration for any n.

Formula of Tchebyshev for n = 2

For $n = 2$, even number, the two nods to be found, x_1 and x_2, with $x_1 = -x_2$, conferred (9), are:

$$u_1 = \frac{n}{6} = \frac{1}{3} = x_1^2 \ \rightarrow x_1 = -\sqrt{u_1} = -\sqrt{\frac{1}{3}} = -0.577350, \ x_2 = 0.577350, \quad \textbf{(49)}$$

and the formula of Tchebyshev for n= 2 is:

$$I = \int_{-1}^{1} f(x)dx \approx f(x_1) + f(x_2) .= f(-0.577350) + f(0.577350) \quad \textbf{(50)}$$

Formula of Tchebyshev for n = 3

Three nods, x_1, x_2 and x_3, with $x_2 = 0$ and $x_1 = -x_3$. We have:

$$u_1 = \frac{n}{6} = \frac{3}{6} = \frac{1}{2} = x_1^2 \ \rightarrow x_1 = -\sqrt{u_1} = -0.707107, \ x_3 = 0.707107;$$

whence the nods of integration :

$$x_1 = -0.707107, \ x_2 = 0, \ x_3 = 0.707107, \quad \textbf{(51)}$$

and the formula of Tchebyshev for n = 3:

$$I = \int_{-1}^{1} f(x)dx \approx f(x_1) + f(x_2) + f(x_3) = f(-0.707107) + f(0) + f(0.707107). \quad \textbf{(52)}$$

Formula of Tchebyshev for n = 4

For $n = 4$, four nods to be found, x_1, x_2, x_3 and x_4, with $x_1 = -x_4$ and $x_2 = -x_3$. Based on (48), the coefficients of (46) are:

$$A_1 = -k_1 = (n{=}4) = -\frac{n}{6} = -\frac{4}{6}, \ A_2 = (s{=}2) = -\frac{1}{2}(k_2 + A_1k_1) = (k_2 = \frac{n}{10} = \frac{4}{10}) = \frac{1}{45}.$$

Thus, equation (46) becomes:

$u^2 - \frac{4}{6} u + \frac{1}{45} = 0,$ **(53)**

whence the nods of integration:

$x_1 = - 0.794654, x_2 = - 0.187592, x_3 = 0.187592, x_4 = 0.794654.$ **(54)**

and the formula of Tchebyshev for n = 4:

$I = \int_{-1}^{1} f(x)dx \approx f(x_1) + f(x_2) + f(x_3) + f(x_4) = f(-0.794654) + f(- 0.187592) + f(0.187592) + f(0.794654)$ **(55)**

Formula of Tchebyshev for n = 5

For n = 5, five nods, the central one being $x_3 = 0$. The remaining nods are x_1, x_2, x_4 and x_5 with $x_1= -x_5$, $x_2= -x_4$. The coefficients of (46) are:

$k_1 = \frac{5}{6} = - A_1; k_2 = \frac{5}{10}, \; A_2 = - \frac{1}{2} (k_2+ A_1k_1) = \frac{7}{72}.$

The equation (46) becomes:

$u^2 - \frac{5}{6} u + \frac{7}{72} = 0,$

whence the nods of integration:

$x_{1,5} = \mp 0.832498, x_3 = 0, x_{2,4} = \mp 0..374541$ **(56)**

and the formula of Tchebyshev for n = 5:

$I = \int_{-1}^{1} f(x)dx \approx f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) = f(-0.832498) + f(-0..374541) + f(0) + f(0..374541) + f(0.832498)$ **(57)**

3.3 Formulas of Gauss

Here, the nods of integration, x_k, and coefficients, C_k, are to be determined under conditions (9) and (10). If the domain of integration is divided into an even number, n, of sub domains, then (7) and (8) for finding x_k and C_k are reduced to the next systems:

$C_1 + C_2 + C_3 + \ldots + C_{\frac{n}{2}} = 1,$

$C_1x_1^2 + C_2 x_2^2 + C_3 x_3^2 + \ldots + C_{\frac{n}{2}} \; x_{\frac{n}{2}}^2 = \frac{1}{3},$ **(58)**

$C_1x_1^4 + C_2 x_2^4 + C_3 x_3^4 + \ldots + C_{\frac{n}{2}} \; x_{\frac{n}{2}}^4 = \frac{1}{5},$

…………………………………………

$C_1x_1^{2(n-1)} + C_2 x_2^{2(n-1)} + C_3 x_3^{2(n-1)} + \ldots + C_{\frac{n}{2}} \; x_{\frac{n}{2}}^{2(n-1)} = \frac{1}{2n-1}.$

But if n is an odd number, the coefficient $C_{\frac{n+1}{2}}$ corresponding to the center of symmetry, $x_{\frac{n+1}{2}} = 0$ should be added to the first equation of (58). Thus, we have the system:

$2(C_1 + C_2 + C_3 + \ldots + C_{\frac{n-1}{2}}) + C_{\frac{n+1}{2}} = 2,$

$C_1x_1^2 + C_2 x_2^2 + C_3 x_3^2 + \ldots + C_{\frac{n-1}{2}} \; x_{\frac{n-1}{2}}^2 = \frac{1}{3},$ **(59)**

$C_1x_1^4 + C_2 x_2^4 + C_3 x_3^4 + \ldots + C_{\frac{n-1}{2}} \; x_{\frac{n-1}{2}}^4 = \frac{1}{5},$

…………………………………………

$C_1x_1^{2(n-1)} + C_2 x_2^{2(n-1)} + C_3 x_3^{2(n-1)} + \ldots + C_{\frac{n-1}{2}} \; x_{\frac{n-1}{2}}^{2(n-1)} = \frac{1}{2n-1}.$

Put $u_k = x_k^2$ as previously done. (58) and (59) respectively become:

$C_1 + C_2 + C_3 + \ldots + C_{\frac{n}{2}} = 1,$

$C_1u_1 + C_2 u_2 + C_3 u_3 + \ldots + C_{\frac{n}{2}} \; u_{\frac{n}{2}} = \frac{1}{3},$ **(60)**

$C_1u_1^2 + C_2 u_2^2 + C_3 u_3^2 + \ldots + C_{\frac{n}{2}} \; u_{\frac{n}{2}}^2 = \frac{1}{5},$

…………………………………………

$C_1u_1^{(n-1)} + C_2 u_2^{(n-1)} + C_3 u_3^{(n-1)} + \ldots + C_{\frac{n}{2}} \; u_{\frac{n}{2}}^{(n-1)} = \frac{1}{2n-1},$

and:

$2(C_1 + C_2 + C_3 + \ldots + C_{\frac{n-1}{2}}) + C_{\frac{n+1}{2}} = 2,$

$C_1u_1 + C_2 u_2 + C_3 u_3 + \ldots + C_{\frac{n-1}{2}} u_{\frac{n-1}{2}} = \frac{1}{3},$ **(61)**

$C_1u_1^2 + C_2 u_2^2 + C_3 u_3^2 + \ldots + C_{\frac{n-1}{2}} u_{\frac{n-1}{2}}^2 = \frac{1}{5},$

…………………………………………

$C_1u_1^{(n-1)} + C_2 u_2^{(n-1)} + C_3 u_3^{(n-1)} + \ldots + C_{\frac{n-1}{2}} u_{\frac{n-1}{2}}^{n-1} = \frac{1}{2n-1}.$

Assume that (60) has $\frac{n}{2}$ rows with the same number of unknowns. As previously indicated, these unknowns are the roots of the equation:

$u^{\frac{n}{2}} + A_1u^{\frac{n}{2}-1} + A_2u^{\frac{n}{2}-2} + \ldots + A_{\frac{n}{2}-1}u + A_{\frac{n}{2}} = 0,$ **(62)**

(for n - even)

$u^{\frac{n-1}{2}} + A_1 u^{\frac{n-1}{2}-1} + A_2 u^{\frac{n-1}{2}-2} + A_3 u^{\frac{n-1}{2}-3} + \ldots + A_{\frac{n-1}{2}-1}u + A_{\frac{n-1}{2}} = 0,$ **(63)**

(for n - odd).

Solving (62) or (63) leads to the nods of integration to be used for finding the coefficients C_k.

Formula of Gauss for n = 2

Two equal coefficients $C_{1,2}$, $C_1 = C_2$; $\frac{n}{2} = 1$, whence the system of equations to be solved for u and C:

$C_1 = 1$ and $C_1u = \frac{1}{3} = u$, $x_{1,2} = \mp\sqrt{\frac{1}{3}} = \mp 0.577350$.

For this case, the formula of Gauss is:

$I = \int_{-1}^{1} f(x)dx \approx C_1f(x_1) + C_2f(x_2) = f(-0.577350) + f(0.577350).$ **(64)**

Formula of Gauss for n = 3

The coefficients are $C_1 = C_3$, C_2, and the nods of integration: x_1, $x_2 = 0$, x_3, $x_1 = -x_3$ The system of equations to be solved for these parameters is:

$2C_1 + C_2 = 2$; $C_1u_1 = \frac{1}{3}$; $C_1u_1^2 = \frac{1}{5}$, $u_1 = \frac{3}{5}$, $x_{1,3} = \mp 0.774597$, $C_1 = C_3 = \frac{5}{9}$, $C_2 = \frac{8}{9}$,

whence the formula of Gauss for n = 3:

$I = \int_{-1}^{1} f(x)dx \approx C_1f(x_1) + C_2f(x_2) + C_3f(x_3) = \frac{5}{9}f(-0.774597) + \frac{8}{9}f(0) + \frac{5}{9}f(0.774597).$ **(65)**

Formula of Gauss for n= 4

The nods are $x_1 = -x_4$, $x_2 = -x_3$ and the coefficients: $C_1 = C_4$ and $C_2 = C_3$. (58) leads to the next system of four equations:

(1) $C_1 + C_2 = 1,$

(2) $C_1u_1 + C_2u_2 = \frac{1}{3},$

(3) $C_1u_1^2 + C_2u_2^2 = \frac{1}{5},$ **(66)**

(4) $C_1u_1^3 + C_2u_2^3 = \frac{1}{7}.$

$n = 4$, $\frac{n}{2} = 2$ - the order of equation (62) to be solved for u_k. This quadratic equation is:

$u^2 + A_1u + A_2 = 0.$ **(67)**

Letting u_1 and u_2 its roots, we may therefore write:

$u_1^2 + A_1u_1 + A_2 = 0,\ u_2^2 + A_1u_2 + A_2 = 0.$ **(68)**

The coefficients A_1 and A_2 of (67) are searched combining the equations of (66) the next way:

(4) + A_1**(3)** + A_2**(2)** and **(3)** + A_1**(2)** + A_2**(1). (69)**

The first combination of (69) leads to:

$C_1u_1^3 + C_2u_2^3 + A_1(C_1u_1^2 + C_2u_2^2) + A_2(C_1u_1 + C_2u_2) = C_1u_1(u_1^2 + A_1u_1 + A_2) + C_2u_2(u_2^2 + A_1u_2 + A_2) = 0 = \frac{1}{7} + \frac{1}{5}A_1 + \frac{1}{3}A_2$,

and the second combination to:

(3) + A_1(2) + A_2(1). = 0 $= \frac{1}{5} + \frac{1}{3}A_1 + A_2$.

Whence the system of the two equations to be solved for A_1 and A_2:

$0 = \frac{1}{7} + \frac{1}{5}A_1 + \frac{1}{3}A_2$, $0 = \frac{1}{5} + \frac{1}{3}A_1 + A_2$. **(70)**

The searched coefficients are $A_1 = -\frac{6}{7}$ and $A_2 = \frac{3}{35}$ and (67) becomes:

$u^2 - \frac{6}{7}u + \frac{3}{35} = 0$ **(71)**

their roots are $u_1 \approx 0.741555 = x_{1,4}^2$; $u_2 \approx 0.115587 = x_{2,3}^2$ Then, we can find the coefficients solving the next system of equations:

$C_1 + C_2 = 1$ and $C_1u_1 + C_2u_2 = 0.741555C_1 + 0.115587C_2 = \frac{1}{3}$

whence $C_1 \approx 0.347855$ and $C_2 \approx 0.652145$.

We have the nods of integration from the root squares of $u_{1,2}$:

$x_{1,4} = \mp\sqrt{u_1} = \mp 0.861136$, $x_{2,3} = \mp\sqrt{u_2} = \mp 0339981$.

Whence the formula of Gauss for n = 4:

$I = \int_{-1}^{1} f(x)dx \approx C_1f(x_1) + C_2f(x_2) + C_2f(x_3) + C_1f(x_4) = 0.347855f(-0.861136) + 0.652145f(-0.339981) + 0.652145f(0.339981) + 0.347855f(0.861136)$ **(72)**

Formula of Gauss for n = 5

The coefficients are $C_1 = C_5$, $C_2 = C_4$, C_3 and the nods of integration: $x_1 = -x_5$, $x_2 = -x_4$, $x_3 = 0$,. The system of the five equations to solve for these parameters is:

(1) $2(C_1 + C_2) + C_3 = 2$; **(2)** $C_1u_1 + C_2u_2 = \frac{1}{3}$; **(3)** $C_1u_1^2 + C_2u_2^2 = \frac{1}{5}$;
(4) $C_1u_1^3 + C_2u_2^3 = \frac{1}{7}$; **(5)** $C_1u_1^4 + C_2u_2^4 = \frac{1}{9}$.

As $\frac{n-1}{2} = 2$, so the quadratic equation to be solved for the nods is:

$u^2 + A_1u + A_2 = 0$. **(73)**

The combinations to form for finding the coefficient of (73) are:

(5) + A_1**(4)** + A_2**(3)** and **(4)** + A_1**(3)** + A_2**(2)**
(6)

which lead to the system of two equations with two unknowns $A_{1,2}$:

$\frac{1}{9} + \frac{1}{7}A_1 + \frac{1}{5}A_2 = 0$ and $\frac{1}{7} + \frac{1}{5}A_1 + \frac{1}{3}A_2 = 0$, **(74)**

Solving (74), (73) becomes:
$u^2 - 1.1111182u + 0.2380980 = 0$

whence $u_1 = 0.8211677 = x_{1,5}^2$, $u_2 = 0.2899506 = x_{2,4}^2$,and the nods $x_{1,5} = \mp 0.906183$, $x_{2,4} = \mp 0.5384706$, $x_3 = 0$.

The coefficients are obtained solving the system of equations:

$C_1u_1 + C_2u_2 = \frac{1}{3}$; $C_1u_1^2 + C_2u_2^2 = \frac{1}{5}$;

whence $C_{1,5} = 0.236922$, $C_{2,4} = 0.478635$, $C_3 = 0.5688856$.

The formula of Gauss for n = 5 is:

$I = \int_{-1}^{1} f(x)dx \approx C_1f(x_1) + C_2f(x_2) + C_3f(x_3) + C_2f(x_4) + C_1f(x_5) = 0.236922f(-0.906183) + 0.478635f(-0.5384706) + 0.5688856f(0) + 0.478635f(0.5384706) + 0.236922f(0.906183)$. **(75)**

4. Estimations of the errors generated by each method

Different estimations of the errors generated by each method, ε_{2m}, are indicated in Table 1. This table indicates that for the three methods and all the considered number of nods, n, the error generated for 2m = 0 is $\varepsilon_0 = 0$. The methods of Newton-Cotes and Tchebyshev have led to the same result for n = 2, 3, 4 and 5 and 2m = 2 (i.e. m = 1). Thus, we conclude that if the integrating function (4) is of first order, these two numerical methods lead to accurate results. Otherwise. Elsewhere the same conclusion can be made when the estimations is $\varepsilon_{2m} = 0$. The error starts to occur when $\varepsilon_{2m} \neq 0$ like ε_4 and ε_6 for n = 2 and 3, indicating that if the order of the function (4) is at least 2 (case of ε_4) or 3 (case of ε_6) the result will not be accurate because of errors generated. In general, for all the three methods the estimations ε_6 (i.e. m = 3) are different from zero meaning that during practical works the best degree of the function (4) to be considered for better results should not reach 3.

Table 1. Table of the estimations of the errors

		Newton-Cotes		Tchebyshev		Gauss	
n	2m	S_{2m}	ε_{2m}	S_{2m}	ε_{2m}	S_{2m}	ε_{2m}
2	0			2	0	2	0
	2			0.667	0	0.667	0
	4			0.222	-0.178	0.222	-0.178
	6			0.074	-0.212	0.074	-0..212
3	0	2	0	2	0	2	0
	2	0.667	0	0.667	0	0.478	-0.189
	4	0.519	0.119	0.333	-0.067	0.148	-0.252
	6	0.502	0.216	0.167	-0.119	0.046	0.003
4	0	2	0	2	0	2	0
	2	0.667	0	0.667	0	0.666	-0.001
	4	0.400	0	0.400	0	0.399	-0.001
	6	0.333	0.048	0.252	-0.034	0.286	0
5	0	2	0	2	0	2	0
	2	0.667	0	0.667	0	0.668	0.001
	4	0.400	0	0.400	0	0.400	0
	6	0.313	0.027	0.269	-0.017	0.143	-0.143

The temptation to develop other formulas for n>5 has indicated that when n increases the coefficients of the equations and systems of equations to be solved for the needed parameters, become very smaller and could probably bring to ill-conditioned problems, (Wilkinson, 1959). Moreover higher orders of these equations and systems of equations complicate the computation process and should also be a source of other errors, between others, round off errors. Table 1 also shows that corresponding estimations of the errors generated by these methods are smaller for the method of Gauss than the ones issued from the method of Tchebyshev and these ones are smaller than the ones from the method of Newton-Cotes. Therefore, it is obvious that the method of Gauss gives more accurate results compared to the methods of Tchebyshev and Newton-Cotes, respectively.

5. Conclusion

Obviously, the numerical methods are very helpful for those function whose anti derivative cannot be found analytically. To avoid round off errors and ill-conditioned problems during practical works, only a few nods of integration, not exceeding 5, should be considered and for more accurate results, integrating function should be replaced by algebraic function of order not exceeding 2.

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