International Journal of Science Academic Research
Vol. 05, Issue 07, pp.7772-7783, July, 2024
Available online at http://www.scienceijsar.com
ISSN: 2582-6425

Research Article

ANALYSIS OF SYMMETRIES, CONSERVATION LAWS, AND EXACT SOLUTIONS OF (1+1) REACTION-DIFFUSION EQUATION

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Received 09th May 2024; **Accepted** 14th June 2024; **Published online** 19th July 2024

Abstract

This research work provides an investigation of the (1+1) reaction-diffusion equation, which models population dynamics with spatially varying growth rates represented by z(x) using Lie point symmetries analysis. Our methodology involves categorising this equation into three distinct types based on the constraints imposed on the spatially dependent growth rate during the solution of the Lie group determining equations. For each category, we systematically derive the corresponding conservation laws associated with the identified symmetries. Additionally, we develop exact solutions for each type, offering a widespread understanding of the population dynamics modelled by the equation. We pay special attention to scale-invariant solutions, which are explored using the global invariants of the one-parameter group. This in-depth investigation not only enhances our theoretical understanding of reaction-diffusion processes in heterogeneous environments but also highlights the utility of symmetry methods in solving complex differential equations.

Keywords: Lie point symmetries, conservation laws, self-adjointness, exact solutions, invariant solutions, and travelling wave solutions

1. INTRODUCTION

Conservation laws [1] are essential in mathematics and physics that explain values that do not change over time and represent the underlying dynamics and symmetries of a system. They represent laws of physics like conservation of mass, energy, and momentum, which state that the overall amount of these attributes does not change when a system changes. The conservation laws are essential to the formulation and analysis of models in mathematics, especially partial differential equations (PDEs), as they provide information on the behaviour and integrability of a system. When a PDE has many conservation laws, it is considered integrable, suggesting that there may be underlying symmetries and allowing for the investigation of exact solutions and other analytical methods. A foundational work in mathematical physics, Noether's theorem [2] reveals a deep relationship between conservation laws and symmetries [3], especially when it comes to Euler-Lagrange equations that are derived from variational principles. This theorem provides a useful tool for deriving conservation laws from the symmetries inherent in the underlying physical or mathematical system. It creates a direct correlation between significant generalised variational symmetries of functional and nontrivial conservation laws. This relationship makes it possible to build conservation laws for Euler-Lagrange equations in a systematic way, as shown in studies [4], clarifying the conservation of significant physical quantities. However, scholars have expanded and generalised Noether's approach to developing conservation laws for systems that do not directly derive from variational principles, including single evolution equations [5-16]. This allows researchers to investigate conservation characteristics even in non-variational situations. These generalisations of Noether's approach give valuable understandings into the conservation features of other mathematical and physical systems other than those derived from variational principles, increasing the applicability of Noether's theorem. Among these generalisations, Ibragimov [5] introduced several key concepts, such as the adjoint equation, strictly self-adjointness, quasi-self-adjointness, and nonlinear self-adjointness, to extend Noether's methodology for identifying conservation laws. He developed an effective technique for determining conservation laws for a combined system containing of the primary equation and its adjoint. This approach allows the derivation of conservation laws using a formal Lagrangian, L, which includes a nonphysical variable, v. When the equation exhibits (nonlinear) self-adjointness, a particular substitution can be employed to eliminate v, yielding the conservation law of our main equation. Conversely, if the equation does not possess self-adjoint properties, the conservation laws derived for every symmetry can be explained as local conservation laws of the combined system. These conservation laws retain the original equation's symmetry characteristics, giving useful information even in the absence of self-adjointness. Following Ibragimov's pioneering work, several scholars have reached into this fascinating field, resulting in an extensive body of literature. Many publications have been published on the subject, including those mentioned in [17-22], as well as countless more relevant studies noted there. These studies have helped to further our understanding of conservation laws for systems defined by partial differential equations, providing new viewpoints, approaches, and applications. The derivations of Lie symmetries, and conservation laws of fractional-order partial differential equations [23] is also very challenging.

The$(1+1)$ reaction-diffusion equation [24] examined in the paper is expressed as:

$$y_t = \rho y_{xx} + z(x)y - y^2, \quad (1)$$

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where $y = y(x,t)$ denotes the population density, $\rho(\neq 0)$ is a constant signifying the diffusion rate, and $z(x)$ is the variable coefficient representing the spatially dependent growth rate of the population. This equation, known as the diffusive logistic equation, models the population dynamics within an environment characterized by spatial heterogeneity. The growth rate $z(x)$ reflects the habitat's spatial variability, being positive in regions conducive to growth and negative in less favourable areas. Cantrell and Cosner [24] explored the impact of this spatial heterogeneity on the modeled population dynamics. When the coefficient $z(x)$ is a constant, denoted as z, the equation simplifies to:

$$y_t = \rho y_{xx} + A(y), \tag{2}$$

where $A(y) = zy - y^2$.This form of the equation is recognised as the nonlinear heat equation [25] when$d = 1$. The nonlinear term $A(y)$ captures the effects of the interaction between diffusion and reaction processes on population dynamics.

Ide and Okada focused on creating numerical schemes of equation (1) while making sure that energy properties are maintained in their work [26]. This is important because, in numerical simulations, particularly in physical systems, energy conservation is frequently a desired property. Their goal was to correctly capture the dynamics of the population given by equation (1) without adding misleading numerical artefacts by creating numerical schemes that preserve energy conservation. To confirm the effectiveness of their proposed numerical systems, they also carried out numerical experiments. To confirm the accuracy and stability of their numerical approaches, these tests probably entailed employing the numerical schemes they established to simulate the population's behaviour over time and comparing the findings with analytical solutions or known system characteristics.

The research begins by addressing the determining equations to identify the infinitesimal generator, which allows the classification of equation (1) into three distinct categories. Following this, the study focuses on deriving the form of the conservation law that corresponds to each identified symmetry generator type. An in-depth analysis of the self-adjointness of equation (1) is conducted to further reveal the nature of these conservation laws. As per the theorem referenced in [27], it is demonstrated that no local conservation law exists for equation (1). Nonetheless, utilising Ibragimov's formula, conservation laws are formulated for each symmetry type. Lastly, the derived symmetry group and an alternative method, exact solutions are established for the final two classifications of equation (1), providing comprehensive insights into its behaviour and solutions.

The organisation of this research is outlined in the following look: Section 2 provides introduction, including a detailed presentation of Ibragimov's theorem. Additionally, it covers foundational concepts and sets the stage for the subsequent analysis by summarising key theoretical frameworks and methodologies relevant to the study. This comprehensive overview ensures that readers are well-prepared for the more technical discussions and findings presented in the later sections of the paper. After that, Section 3 begins by using a powerful computational method [4] to solve equation (1) to determine its Lie point symmetries while also investigating its self-adjointness. The emphasis of Section 4 is to provide an in-depth review of the conservation laws that are part of equation (1). The derivation of many exact solutions for equation (1) is the focus of Section 5. Section 6 concludes the report by providing a summary of the main conclusions and implications of the studies.

2. PRELIMINARIES

We rush back Ibragimov's approach for deriving conservation laws associated with the symmetries of any system of partial differential equations (PDEs), assuming the system has an equal number of equations and dependent variables. For simplicity, we focus on a scalar evolution equation:

$$F\left(x, y, y_{(1)}, \dots, y_{(r)}\right) = 0. \tag{3}$$

Here,$x = (x^1, t)$, and the dependent variable is y. The notation $y_{(1)}$ represents the set of first-order partial derivatives y_i, while $y_{(2)}$ represents the set of second-order partial derivatives y_{ij}, and so on, with $y_i = \partial y/\partial x^i$ and $y_{ij} = \partial^2 y/\partial x^i \partial x^j$, etc.

Definition 2.1 *The adjoint equation corresponding to (3) is defined as:*

$$F^*\left(x, y, v, y_{(1)}, v_{(1)}, \dots, y_{(r)}, v_{(r)}\right) = 0, \tag{4}$$

where

$$F^*\left(x, y, v, y_{(1)}, v_{(1)}, \dots, y_{(r)}, v_{(r)}\right) = \frac{\delta(vF)}{\delta y}, \tag{5}$$

with $v = v(x^1, t)$ serving as a multiplier.
The variational derivatives, denoted by

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} + \sum_{r=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_r} \frac{\partial}{\partial y_{i_1 \cdots i_r}}, \tag{6}$$

represent the Euler-Lagrange operator, where

$$D_i = \frac{\partial}{\partial x^i} + y_i \frac{\partial}{\partial y} + y_{ij} \frac{\partial}{\partial y_j} + \cdots, \tag{7}$$

accounts for total differentiation. We apply equation (3) to a system in the following way:

$$\frac{\delta(vF)}{\delta v} = F\left(x, y, y_{(1)}, \dots, y_{(r)}\right) = 0,$$
$$\frac{\delta(vF)}{\delta y} = F^*\left(x, y, v, y_{(1)}, v_{(1)}, \dots, y_{(r)}, v_{(r)}\right) = 0. \tag{8}$$

Ibragimov [15] demonstrated that the system (8) preserves all symmetries of the original equation (3). Using Noether's identity [15], he developed a formula for the conservation law of each symmetry of (3). This method generates conservation rules for PDEs in a systematic manner by utilising their symmetries and adjoint systems.

Theorem 2.1 *Each Lie point, Lie-Backlund, or nonlocal symmetry,*

$$X = \xi^i\left(x, y, y_{(1)}, \dots\right) \frac{\partial}{\partial x^i} + \eta\left(x, y, y_{(1)}, \dots\right) \frac{\partial}{\partial y}, \tag{9}$$

of (3) yields a conservation law: $D_i(\mathcal{T}^i) = 0$ for (8). The conserved vector is formulated as

$$\mathcal{T}^i = \xi^i L \tag{10}$$
$$+ w\left[\frac{\partial L}{\partial y_i} - D_j\left(\frac{\partial L}{\partial y_{ij}}\right) + D_j D_k\left(\frac{\partial L}{\partial y_{ijk}}\right) - \cdots\right]$$
$$+ D_j(w)\left[\frac{\partial L}{\partial y_{ij}} - D_k\left(\frac{\partial L}{\partial y_{ijk}}\right) + D_k D_r\left(\frac{\partial L}{\partial y_{ijkr}}\right) - \cdots\right]$$
$$+ D_j D_k(w)\left[\frac{\partial L}{\partial y_{ijk}} - D_r\left(\frac{\partial L}{\partial y_{ijkr}}\right) + \cdots\right] + \cdots,$$

in *Theorem 2.1*, w and L are written in the following form:

$$w = \eta - \xi^j y_j, \quad L = vF\left(x, y, y_{(1)}, \dots, y_{(r)}\right). \tag{11}$$

Our main equation (1) is of the second-order equation, so (10) can be written as:

$$\mathcal{T}^i = \xi^i L + w\left[\frac{\partial L}{\partial y_i} - D_j\left(\frac{\partial L}{\partial y_{ij}}\right)\right] + D_j(w)\frac{\partial L}{\partial y_{ij}}. \tag{12}$$

It is clear from *Theorem 2.1* that any symmetry in equation (3) has the capacity to produce a conservation law. These conservation laws usually depend on the multiplier v in addition to the original variables of the equation. In some cases, the multiplier v can be eliminated from the conservation law if the equation (3) has the self-adjointness feature.

Definition 2.2 *Equation (3) is defined as self-adjoint if, when the adjoint equation (4) is transformed by substituting* $v = u$, *the resulting equation*

$$F\left(x, y, v, y_{(1)}, v_{(1)}, \dots, y_{(r)}, v_{(r)}\right) = 0, \tag{13}$$

is same as our main equation (3). This condition implies that

$$F\left(x, y, v, y_{(1)}, v_{(1)}, \dots, y_{(r)}, v_{(r)}\right)\big| v = y \tag{14}$$
$$= \eta\left(x, y, y_{(1)}, \dots\right) F\left(x, y, y_{(1)}, \dots, y_{(r)}\right),$$

where η is a function that depends on x, y, and the partial derivatives of y.

Definition 2.3 *Equation (3) is termed quasi-self-adjoint if the adjoint equation (4), transformed under the substitutions* $v = \psi(y)$ *where* $\psi(y)$ *is a specific function with* $\psi'(y) \neq 0$, *leads to an equation*

$$F^*\left(x, y, v, y_{(1)}, v_{(1)}, \dots, y_{(r)}, v_{(r)}\right) = 0, \tag{15}$$

that exactly matches the original equation (3).

Definition 2.4 *Equation (3) is classified as weakly self-adjoint if, upon substituting* $v = \psi(x, y)$ *into the adjoint equation (4), where* $\psi(x, y)$ *is a function such that* $\psi_y \neq 0$ *and* $\psi_x \neq 0$*, the resulting equation*

$$F^*\left(x, y, v, y_{(1)}, v_{(1)}, \dots, y_{(r)}, v_{(r)}\right) = 0, \tag{16}$$

is look like our equation (3).

This means the transformed adjoint equation retains the equal form as the initial equation, ensuring that the properties of the original equation are preserved under the specified substitutions.

Definition 2.5 *Equation (3) exhibits nonlinear self-adjointness when there are functions* $v = \psi(x, y)$ *that satisfy the adjoint equation (4) for all solutions* $y(x)$ *of (3), alongside the condition that* $\psi(x, y) \neq 0$.

Ibragimov in [5, 11, 28] proposed the initial three definitions, respectively, while the fourth was initially presented by Gandarias in [29]. Subsequently, Ibragimov extended this definition in [15] under the appearance of *Definition 2.5*. Hence, nonlinear self-adjointness emerges as the broadest concept, encompassing the others as mere special cases.

Theorem 2.2 *The form that characterised any local conservation law [27] relevant to a second order (1+1)-dimensional quasilinear evolution equation is written as:*

$$y_t = S(x, t, y, y_x)y_{xx} + A(x, t, y, y_x), \tag{17}$$

where $S(x, t, y, y_x) \neq 0$, is necessarily of the first order.

Furthermore, it is possible to identify a conserved vector composed of a density T, which is contingent upon t, x, and y at most, and a flux X, dependent on t, x, y, and y_x at most.

3. ANALYSIS OF SELF-ADJOINTNESS AND DETERMINING LIE POINT SYMMETRIES

Here, we work on the self-adjointness of equation (1) and develop a generic Lie group of point transformations that maintains equation (1) invariant.

The following is the shape of our one-parameter Lie group of infinitesimal transformations:

$$\begin{aligned} x &\longrightarrow x + \varepsilon\xi^1(x, t, y), \\ t &\longrightarrow t + \varepsilon\xi^2(x, t, y), \\ y &\longmapsto u + \varepsilon\eta(x, t, y), \end{aligned} \tag{18}$$

here $\varepsilon \ll 1$. We define the symmetry generator for the group transformations (18) , which takes the following look:

$$\mathcal{X} = \xi^1(x, t, y)\frac{\partial}{\partial x} + \xi^2(x, t, y)\frac{\partial}{\partial t} + \eta(x, t, y)\frac{\partial}{\partial y}. \tag{19}$$

The highest order of equation (1) is two, so we apply the second prolongation

$$pr^{(2)}\mathcal{X} = \mathcal{X} + \eta_x\frac{\partial}{\partial y_x} + \eta_t\frac{\partial}{\partial y_t} + \eta_{xx}\frac{\partial}{\partial y_{xx}} + \eta_{xt}\frac{\partial}{\partial y_{xt}} + \eta_{tt}\frac{\partial}{\partial y_{tt}}. \tag{20}$$

The coefficient functions $\xi^1(x, t, y), \xi^2(x, t, y)$, and $\eta(x, t, y)$ will satisfy the symmetry condition which the defined as:

$$\eta_t - d\eta_{xx} - z_x(x)y\xi^1(x, t, y) - z(x)\eta(x, t, y) + 2y\eta(x, t, y) = 0, \tag{21}$$

we can observe that η_t, η_{xx} are the coefficients which appear in prolongation equation (20) and we get the following system:

$$\begin{aligned} \eta_t = \quad & D_t\eta(x, t, y) - y_x D_t\xi^1(x, t, y) - y_t D_t\xi^2(x, t, y), \\ \eta_{xx} = \quad & D_x^2\eta(x, t, y) - y_x D_x^2\xi^1(x, t, y) - y_t D_x^2\xi^2(x, t, y) \\ & -2y_{xx}D_x\xi^1(x, t, y) - 2y_{xt}D_x\xi^2(x, t, y), \end{aligned} \tag{22}$$

D_x, D_t with respect to x and t can be found using equation (7) which are the The total derivatives.

Substitute system of equations (22) into equation (21) and replace y_t by $dy_{xx} + z(x)y - y^2$. We find determining equations by equating the coefficients of the first- and second-order partial derivatives of y:

$$\xi^1_y = 0, \tag{23}$$

$$\xi^2_y = 0, \tag{24}$$

$$\eta_{yy} = 0, \tag{25}$$

$$\xi^2_x = 0, \tag{26}$$

$$-\rho\xi^2_t + 2\rho\xi^1_x = 0, \tag{27}$$

$$-\xi^1_t - 2\rho\eta_{xy} + \rho\xi^1_{xx} = 0, \tag{28}$$

$$(-z(x) + 2y)\eta + y\xi^1 z_x(x) + (-y^2 + yz(x))\eta_y + \eta_t + (y^2 - yz(x))\xi^2_t - \rho\eta_{xx} = 0. \tag{29}$$

We find the solution from equation (23)-equation (27) very easily:

$$\begin{aligned} \xi^2(x,t,y) = &\quad \xi^2(t), \\ \eta(x,t,y) = &\quad F^2(x,t)y + F^3(x,t), \\ \xi^1(x,t,y) = &\quad \tfrac{1}{2}\xi^2_t(t)x + F^4(t), \end{aligned} \tag{30}$$

for some unknown functions $F^3(x,t), F^2(x,t)$, and $F^4(t)$. We substitute equation (30) in the equations (28) and equation (29), which gives us the following system:

$$-\xi^1_t(x,t) - 2\rho F^2_x(x,t) = 0, \tag{31}$$

$$\begin{aligned} -z(x)(F^2(x,t)y \quad &+F^3(x,t)) - z_x(x)y\left(\tfrac{1}{2}\xi^2_t(t)x + F^4(t)\right) + yz(x)F^2(x,t) \\ &-yz(x)\xi^2_t(t) + y^2(F^2(x,t) + \xi^2_t(t)) + y(2F^3(x,t) + F^2_t(x,t) \\ &-\rho F^2_{xx}(x,t)) + F^3_t(x,t) - \rho F^3_{xx}(x,t) = 0. \end{aligned} \tag{32}$$

We know that $z(x)$ is not known; we break equation (32) into such a way that a part contains $z(x)$ and $z_x(x)$ and other without them:

$$-z(x)F^3(x,t) - z_x(x)y\left(\tfrac{1}{2}\xi^2_t(t)x + F^4(t)\right) - yz(x)\xi^2_t(t) = 0, \tag{33}$$

$$y^2(F^2(x,t) + \xi^2_t(t)) + y(2F^3(x,t) + F^2_t(x,t) - \rho F^2_{xx}(x,t)) + F^3_t(x,t) - \rho F^3_{xx}(x,t) = 0. \tag{34}$$

By rearranging equation (34) in the following way:

$$y\left[-z(x)\xi^2_t(t) - z_x(x)\left(\tfrac{1}{2}\xi^2_t(t)x + F^4(t)\right)\right] - z(x)F^3(x,t) = 0. \tag{35}$$

We compare the coefficients of y and write the equations:

$$\begin{aligned} -z(x)\xi^2_t(t) - z_x(x)\left(\tfrac{1}{2}\xi^2_t(t)x + F^4(t)\right) = 0, \\ -z(x)F^3(x,t) = 0. \end{aligned} \tag{36}$$

Again, compare the coefficients of y^2, y and constant in equation (34).

$$F^2(x,t) + \xi^2_t(t) = 0, \tag{37}$$

$$2F^3(x,t) + F^2_t(x,t) - \rho F^2_{xx}(x,t) = 0, \tag{38}$$

$$F^3_t(x,t) - \rho F^3_{xx}(x,t) = 0. \tag{39}$$

From equation (37)

$$F^2(x,t) = -\xi^2_t(t),\ \ F^2_x(x,t) = F^2_{xx}(x,t) = 0, \tag{40}$$

Equation (38) provides

$$F^3_x(x,t) = F^3_{xx}(x,t) = 0, \tag{41}$$

We find that equation (39) is

$$F_t^3(x,t) = 0, \tag{42}$$

and $F^3(x,t)$ is a constant. We obtain that equation (31) into reduced form which is the the following:

$$-\frac{1}{2}\xi_{tt}^2(t)x - F_t^4(t) = 0. \tag{43}$$

We compare the coefficients of x and constant in equation (43), which give us $\xi_{tt}^2(t) = 0, F_t^4(t) = 0$, and we find

$$\xi^2(t) = q_1 t + q_2, F^4(t) = q_3, \tag{44}$$

where q_1, q_2 and q_3 are arbitrary constants. We get $F^3(x,t) = 0$ and the general solution of the system (31):

$$\begin{aligned} \xi^1(x,t,y) &= \frac{1}{2}q_1 x + q_3, \\ \xi^2(x,t,y) &= q_1 t + q_2, \\ \eta(x,t,y) &= -q_1 y. \end{aligned} \tag{45}$$

We investigate equation (35) and substitute all the required values, and we find

$$\left(\frac{1}{2}q_1 x + q_3\right) z_x(x) + q_1 z(x) = 0. \tag{46}$$

By dealing with arbitrary constants q_i, we can classify equation (1) in the following three types:

Case 1.

When $q_2 = 1, q_1 = q_3 = 0, z(x)$ will remain free function and equation (1) Will have the following infinitesimal generator:

$$\mathcal{X}_1 = \partial_t. \tag{47}$$

So, equation (1) does not depend on t.

Case 2.

When $q_1 = 1, q_2 = q_3 = 0$, $\frac{1}{2}xz_x(x) + z(x) = 0$ and solve for $z(x)$, which is $z(x) = \frac{q_4}{x^2}$, where q_4 is known to be an integral constant and the infinitesimal generator provides the following symmetry:

$$\mathcal{X}_2 = \frac{1}{2}x\partial_x + t\partial_t - y\partial_y. \tag{48}$$

Our main equation (1) gets the following new form:

$$y_t - \rho y_{xx} - \frac{q_4}{x^2}y + y^2 = 0, \tag{49}$$

having the infinitesimal generators $\mathcal{X}_1, \mathcal{X}_2$. We build a table of the commutation of the Lie algebra generated by using the infinite symmetries, represented by Table 1:

Table 1: The Commutator Table.

$[\cdot,\cdot]$	$\mathcal{X}_1$	$\mathcal{X}_3$
$\mathcal{X}_1$	0	0
$\mathcal{X}_3$	0	0

Case 3.

When $q_3 = 1, q_1 = q_2 = 0, z(x) = z$, where z is a constant, and the infinitesimal generator is gives the following third symmetry:

$$\mathcal{X}_3 = \partial_x. \tag{50}$$

Equation (1) will be of the following form:

$$y_t - \rho y_{xx} - zy + y^2 = 0, \tag{51}$$

with two infinitesimal generators $\mathcal{X}_1, \mathcal{X}_3$. The commutator table of the Lie algebra obtained by the infinite symmetries is:

Table 2: The Commutator Table.

$[\cdot,\cdot]$	$\mathcal{X}_1$	$\mathcal{X}_3$
$\mathcal{X}_1$	0	0
$\mathcal{X}_3$	0	0

We define the adjoint equation for equation (1)

$$\mathcal{A}^* = \frac{\delta}{\delta y}[v(x,t)\mathcal{A}] = -vz(x) + 2vy - v_t - \rho v_{xx} = 0. \tag{52}$$

For $\rho \neq 0$, equation (1) is not to be a strictly self-adjoint, so we set $v = \Upsilon(y)$ in (52).

$$\mathcal{A}^*|_{v=\Upsilon(y)} = -\Upsilon(y)z(x) + 2y\Upsilon(y) -\Upsilon_y(y)y_t - \rho\Upsilon_{yy}(y)y_x^2 - \rho\Upsilon_y(y)y_{xx}. \tag{53}$$

We use equation (14), which gives

$$-\mathcal{A}^*|_{v=\Upsilon(y)} + \Lambda\mathcal{A} = y_t\big(\Lambda + \Upsilon_y(y)\big) + \rho y_{xx}\big(-\Lambda + \Upsilon_y(y)\big) -\Lambda y z(x) + \Lambda y^2 + \rho\Upsilon_{yy}(y)y_x^2 + \Upsilon(y)z(x) - 2y\Upsilon(y) = 0. \tag{54}$$

If we compare the coefficients of y_t and y_{xx}, we get

$$\Lambda + \Upsilon_y(y) = 0, \text{and } -\Lambda + \Upsilon_y(y) = 0. \tag{55}$$

It means that equation (1) is not quasi-self-adjoint, when $d \neq 0$.
Again, we set $v = \Upsilon(x,t,y)$, and we investigate

$$\mathcal{A}^*|_{v=h(x,t,y)} = \Lambda\big(x,t,y,y_{(1)},\dots\big)\mathcal{A}, \tag{56}$$

with total derivatives

$$\begin{aligned} v_t = D_t[h(x,t,y)] &= \Upsilon_y(x,t,y)\cdot y_t + \Upsilon_t(x,t,y),\\ v_x = D_x[h(x,t,y)] &= \Upsilon_y(x,t,y)\cdot y_x + \Upsilon_x(x,t,y),\\ v_{xx} = D_x(v_x) &= \Upsilon_y(x,t,y)\cdot y_{xx} + \Upsilon_{yy}(x,t,y)\cdot y_x^2\\ &\quad +2\Upsilon_{xy}(x,t,y)\cdot y_x + \Upsilon_{xx}(x,t,y). \end{aligned} \tag{57}$$

Substitute equation (57), $v = \Upsilon(x,t,y)$ in equation (52) and compare the coefficients of derivatives terms of y, we find

$$\Lambda = \Upsilon_y(x,t,y) = 0, \tag{58}$$

$$-\Upsilon(x,t,y)z(x) + 2\Upsilon(x,t,y)y - \Upsilon_t(x,t,y) - \rho\Upsilon_{xx}(x,t,y) = 0. \tag{59}$$

Equation (59) needs to hold true for all values of t, x, and y. Upon eliminating the coefficient of y we obtai $\Upsilon(x,t,y)$ =0. Therefore, based on *Definition 2.5*, when $\rho \neq 0$, equation (1) does not exhibit quasi-self-adjointness and nonlinear self-adjointness.

4. CONSERVATION LAWS

This section, we find conserved vectors $(\mathcal{T}^1, \mathcal{T}^2)$, by applying *Theorem 2.2* on equation (1) using equation (52) by using the Lie point symmetries, which fulfil the conservation equation

$$(D_t\mathcal{T}^1 + D_x\mathcal{T}^2)|_{\mathcal{A}=0,\mathcal{A}^*=0} = 0. \tag{60}$$

We have three infinitesimal generators, so there are three different cases through which we find conserved vectors.

Case 1

Equation (12) helps to yield the following conserved vectors when we have $w = -y_t$ for $\mathcal{X}_1$.

$$\mathcal{T}^1 = v[-\rho y_{xx} - z(x)y + y^2], \quad \mathcal{T}^2 = -\rho y_t v_x + \rho v y_{tx}. \tag{61}$$

There is an arbitrary solution v of equation (52) in these conserved vectors and with the help of Mathematica, we obtain:

$$D_t\mathcal{T}^1 + D_x\mathcal{T}^2 = y_t[-\rho v_{xx} - v_t - z(x)v + 2yv]. \tag{62}$$

Case 2

Here, for $\mathcal{X}_2$ we get nontrivial conserved vectors when we have $w = -y - \frac{1}{2}xy_x - ty_t$

$$\begin{aligned} \mathcal{T}^1 = \;& v\left(ty^2 - t\tfrac{q_4}{x^2}y - y - \tfrac{1}{2}xy_x - t\rho y_{xx}\right), \\ \mathcal{T}^2 = \;& \tfrac{1}{2}xv\left(y_t - \tfrac{q_4}{x^2}y + y^2\right) \\ & -\rho v_x\left(y + \tfrac{1}{2}xy_x + ty_t\right) + \rho v\left(\tfrac{3}{2}y_x + ty_{tx}\right). \end{aligned} \tag{63}$$

Like case 1, conserved vectors in this case contain an arbitrary solution v to equation (52), and using Mathematica, we find the following equation:

$$\begin{aligned} D_t\mathcal{T}^1 \;& + D_x\mathcal{T}^2 \\ = \;& \left(y + \tfrac{1}{2}xy_x + ty_t\right)\left(-\rho v_{xx} - v_t - \tfrac{q_4}{x^2}v + 2yv\right) \\ & -\tfrac{1}{2}v\left(y_t - \rho y_{xx} - \tfrac{q_4}{x^2}y + y^2\right). \end{aligned} \tag{64}$$

Case 3

The symmetry $\mathcal{X}_3$ when we have $w = -y_x$ provides nontrivial conserved vectors with the help of equation (12).

$$\begin{aligned} \mathcal{T}^1 &= -y_x v \\ \mathcal{T}^2 &= v(y_t - by + y^2) - \rho y_x v_x. \end{aligned} \tag{65}$$

There is an arbitrary solution v to equation (52), and we use Mathematica software, and find

$$D_t\mathcal{T}^1 + D_x\mathcal{T}^2 = y_x(-\rho v_{xx} - v_t - bv + 2yv). \tag{66}$$

The conserved vectors that we discovered are exclusive to equation (1) and equation (53), but they are not locally relevant to equation (1) on their own. It follows that equation (1) does not contain a local conservation law.

There is the density $\mathcal{P} = \mathcal{P}(x,t,y)$ and the flux $\mathcal{F} = \mathcal{F}\ell(x,t,y,y_x)$ for equation (1) in view of *Theorem 2.2*.

$$(D_t\mathcal{T} + D_x\mathcal{F})|_{y_t=\rho y_{xx}+y(z(x)-y)} = 0, \tag{67}$$

and we get

$$\begin{aligned} & \mathcal{T}_t + \mathcal{T}_y[\rho y_{xx} + y(z(x) - y)] \\ & +\mathcal{F}_x + \mathcal{F}_y y_x + \mathcal{F}_{y_x} y_{xx} = 0. \end{aligned} \tag{68}$$

We get the coefficient of y_{xx}.

$$\rho\mathcal{T}_y(x,t,y) + \mathcal{F}_{y_x}(x,t,y,y_x) = 0; \tag{69}$$

therefore

$$\mathcal{F}(x,t,y,y_x) = -\rho\mathcal{T}_y(x,t,y)y_x + \hat{\mathcal{F}}(x,t,y). \tag{70}$$

Comparing the coefficients of the powers of y_x in the rest of (68), on the functions $\mathcal{T}(x,t,y)$ and $\hat{\mathcal{F}}(x,t,y)$, we derive the system of PDEs.

$$\begin{aligned} & y_x: \quad -\rho\mathcal{T}_{yx}(x,t,y) + \hat{\mathcal{F}}_y(x,t,y) = 0, \\ & y_x^2 \quad : -\rho\mathcal{T}_{yy}(x,t,y) = 0, \\ & 1: \quad \mathcal{T}_t(x,t,y) + \mathcal{T}_y(x,t,y)y(z(x) - y) + \hat{\mathcal{F}}_x(x,t,y) = 0. \end{aligned} \tag{71}$$

In consideration of (71), we suppose the following equation for $\rho \neq 0$:

$$\mathcal{T}(x,t,y)=\eta(x,t)y+\mathcal{T}^0(x,t), \tag{72}$$

and

$$\hat{\mathcal{F}}_y(x,t,y)=\rho\eta_x(x,t), \tag{73}$$

so

$$\hat{\mathcal{F}}(x,t,y)=\rho\eta_x(x,t)y+\mathcal{F}^0(x,t). \tag{74}$$

Now

$$\mathcal{F}(x,t,y,y_x)=-\rho\eta(x,t)y_x+\rho\eta_x(x,t)y+\mathcal{F}^0(x,t). \tag{75}$$

Equations (72) and (75) are substituted into third equation of system (71)

$$\begin{aligned}&\eta_t(x,t)y+\mathcal{T}_t^0(x,t)+\eta(x,t)y(z(x)-y)\\&+\rho\eta_{xx}(x,t)y+\mathcal{F}_x^0(x,t)=0.\end{aligned} \tag{76}$$

The reason we say $\mathcal{T}^0(x,t)=\mathcal{F}^0(x,t)=0$ is that they only add to the conservation law's trivial section. The system of PDEs on $\eta(x,t)$ is obtained by splitting equation (76) regarding the powers of y

$$\begin{aligned}&y:\eta_t(x,t)+\rho\eta_{xx}(x,t)+z(x)\eta(x,t)=0,\\&y^2:-\eta(x,t)=0.\end{aligned} \tag{77}$$

$\eta(x,t)=0$ is the solution for this system, according to equation (77). This indicates that there are no local conservation laws in equation (1).

5. EXACT SOLUTIONS

For equation (49), we obtain scale-invariant solutions, and for equation (51), we find out travelling wave solutions.

Scale-Invariant Solution

We consider

$$\mathcal{X}_2=\frac{1}{2}x\partial_x+t\partial_t-y\partial_y, \tag{78}$$

We define a corresponding scaling group

$$(x,t,y)\mapsto\left(\Lambda^{1/2}x,\Lambda t,\Lambda^{-1}y\right),\Lambda\in\mathbb{R}^+. \tag{79}$$

The functions on the half space $\{(x,t,y),t>0\}$ related to this one $-$ parameter group give the global invariants.

$$\gamma=t^{\left(-\frac{1}{2}\right)}x,\quad v=ty; \tag{80}$$

then

$$\begin{aligned}y&=t^{-1}v,\\y_x&=t^{-1}v_\gamma,\\y_{xx}&=t^{-2}v_{\gamma\gamma},\\y_t&=-t^{-2}\left(v+\frac{1}{2}\gamma v_\gamma\right).\end{aligned} \tag{81}$$

Substitute equation (81) into (48) and we get:

$$-t^{-2}\left(v+\frac{1}{2}\gamma v_\gamma\right)=t^{-2}\left(\rho v_{\gamma\gamma}+q_4\gamma^{-2}v+v^2\right). \tag{82}$$

The variable t does not appear in equation (82) as a parameter.

$$\rho v_{\gamma\gamma}+\frac{1}{2}\gamma v_\gamma+v+q_4\gamma^{-2}v-v^2=0, \tag{83}$$

and the scale-invariant solutions are expressed in the reduced form.
We reduce (83) to the first-order equation when we substitute $v'(\gamma)=\eta(v)$.

$$\rho\eta(v)\eta'(v)+\frac{1}{2}\gamma\eta(v)+v+q_4\gamma^{-2}v-v^2=0. \tag{84}$$

The generic form of equation (84) is

$$(\gamma(x)+g(x))\gamma'(x)=f_2(x)\gamma(x)^2+f_1(x)\gamma(x), \tag{85}$$

where $f_2(x), f_1(x)$ are arbitrary functions because this equation is identical to class A of Abel's second kind. As stated in [30], this ordinary differential equation does not currently have a general solution.

Travelling Wave Solutions

We have a special case when $d=1$.

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}+aw+z_1w^m, \tag{86}$$

We put $a=z, z_1=1, m=2$ using formulation defined in [31], and we get travelling wave solutions

$$y(x,t)=[F^2+\mathcal{C}\exp(\Lambda t\pm\mu x)]^{-2}, \tag{87}$$

$$y(x,t)=[-F^2+\mathcal{C}\exp(\Lambda t\pm\mu x)]^{-2},$$

where $\Lambda=-\frac{5}{6}z, \mu=\sqrt{\frac{z}{6}}$, and $F^2=\sqrt{\frac{1}{z}}$. We verified these solutions on Maple by using pdetest command. We define translation group when $\rho\neq 1$ for equation (51).

$$(x,t,y)\longmapsto(x+c\varepsilon, t+\varepsilon, y),\quad \varepsilon\in\mathbb{R}. \tag{88}$$

This found by using the formula $\partial_t+c\partial_x$, where c is a fixed constant that controls the waves' speed. Consequently, this group's global invariants are

$$\gamma=x-ct,\quad v=y. \tag{89}$$

The appearance of $v=\Upsilon(\gamma)$ is as follows: A wave with an unchanged profile travelling at a constant speed c is found using the $y=\Upsilon(x-ct)$. We solve derivatives by taking the derivative of equation (88) w.r.tx and t in terms of those of vw.r.ty, and find

$$y_t=-cv_\gamma, y_x=v_\gamma, y_{xx}=v_{\gamma\gamma}. \tag{90}$$

After substituting these equations, we get ODE for the travelling wave solution

$$\rho v_{\gamma\gamma}+cv_\gamma+bv-v^2=0. \tag{91}$$

We can find the travelling wave solution of equation (51) if we get the solution of equation (91). We can easily observe its invariance under the group translations in the γ direction.

$$(\gamma,v)\longmapsto(\gamma+\varepsilon,v), \tag{92}$$

with infinitesimal generator $\mathcal{X}=\partial_\gamma$. We get canonical variables

$$\xi^2=\xi^2(\gamma,v),\quad w=w(\gamma,v), \tag{93}$$

by solving the equation (93)

$$\frac{\partial\xi^2}{\partial\gamma}=0, \frac{\partial w}{\partial\gamma}=1 \tag{94}$$

and we find

$$\xi^2(\gamma,v)=v,\quad w(\gamma,v)=\gamma. \tag{95}$$

Then

$$\frac{dv}{d\gamma}=\frac{1}{w_{\xi^2}}, \frac{d^2v}{\rho\gamma^2}=-\frac{w_{\xi^2\xi^2}}{w_{\xi^2}^3}, \tag{96}$$

so equation (91) becomes

$$\rho\left(-\frac{w_{\xi^2\xi^2}}{w_{\xi^2}^3}\right)+c\frac{1}{w_{\xi^2}}+z\xi^2-(\xi^2)^2=0, \tag{97}$$

which is a first-order equation for $\varrho = w_{\xi^2}$:

$$\rho\left(-\frac{\varrho_{\xi^2}}{\varrho^3}\right)+c\frac{1}{\varrho}+z\xi^2-(\xi^2)^2=0 \tag{98}$$

that is

$$\rho\varrho_{\xi^2}=(z\xi^2-(\xi^2)^2)\varrho^3+c\varrho^2. \tag{99}$$

When we set $\varrho=\frac{1}{\omega}$, we get

$$\omega\frac{d\omega}{d\xi^2}=-\frac{1}{\rho}(z\xi^2-(\xi^2)^2)-\frac{c}{\rho}\omega. \tag{100}$$

Substitute $\xi^1=-\frac{c}{\rho}\xi^2$, and equation (100) takes the following form:

$$\omega\frac{d\omega}{d\xi^1}=\omega+\left(-\frac{z\rho}{c^2}\xi^1-\frac{\rho^2}{c^3}(\xi^1)^2\right). \tag{101}$$

If $\frac{z\rho}{c^2}=\pm\frac{6}{25}$ defined in [32], we can find the solution of equation (101) by using the elliptic Weierstrase function. If $\frac{z\rho}{c^2}=\frac{6}{25}$, the solution described the parametric ally in this form:

$$\xi^1=5ar^2\psi,\quad \omega=ar^2\mathcal{E}_1, \tag{102}$$

$\mathcal{E}_1=r\sqrt{\pm(4\psi^3-1)}+2\psi$, where q_2 is a constant, and $a=-\frac{125\rho^2}{6c^3},\quad r=\int\left(\frac{\rho\psi}{\pm\sqrt{4\psi^3-1}}\right)-q_2,\psi$ is known as the classical elliptic We ierstrase function $\psi=\psi(r+q_2,0,1)$.

We write the solutions parametrically in the following form when $\frac{z\rho}{c^2}=-\frac{6}{25}$:

$$\xi^1=5a\mathcal{E}_2,\quad \omega=ar^2\mathcal{E}_1, \tag{103}$$

with $\mathcal{E}_2=r^2\psi\mp 1$.

6. CONCLUSION

We presented several exact solutions to a variable-coefficient (1+1) reaction-diffusion equation, as well as local conservation laws. We began our breakdown by doing Lie symmetry analysis, which allowed us to categorise the problem into three different categories. Every category has unique Lie point symmetries. We successfully acquired the relevant conservation laws for each classified type and expressed them in terms of their respective Lie point symmetries. Our investigation on the solutions of equation (49) that are scale-invariant, also use global invariants associated to the one-parameter group $\mathcal{X}_2$. The scale-invariant solution of the equation (83) should already lead to a constraint on any solution of this type for (49). Finally, many exact solutions of equation (51) are found proving the flexibility and strength: our method can be used to solve various iterations of reaction-diffusion equations. This study not only advances our understanding of a variable-coefficient (1+1) reaction-diffusion equations by offering exact solutions and conservation laws, but it also demonstrates the use of Lie symmetry analysis in the categorization and resolution of complex differential equations.

Conflict of Interests: All the authors agreed on the statement that there is no conflict of interest in the publishing of this research.

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